

Massachusetts Institute of Technology
Instrumentation Laboratory
Cambridge, Massachusetts

COLOSSUS Memo #163, LUMINARY Memo #70

TO: Distribution
FROM: J. A. Saponaro
DATE: March 12, 1969
SUBJECT: Accuracy of V83 (Range, Range Rate, Theta Display)

1.0 Introduction

The purpose of this memo is to provide information on the accuracy of the V83 and V85 range, range rate, and angle displays generated by the SUNDANCE, LUMINARY or COLOSSUS programs and is in response to Mission D Action Item #9. Since these displays are used for backup chart solutions for rendezvous maneuvers and possibly for terminal braking maneuvers for docking, it is important that the reliability and validity of these displays be understood.

The contents of this memo apply to all previously released and current revisions of COLOSSUS, LUMINARY and SUNDANCE programs.

2.0 Summary of Conclusions

For those that like the short of it and generally skip to the last chapter before the first, the following is an outline summary of the main points of this memo.

1. The validity of the range and range display depends on the quality of the on-board states and the numerical accuracy of the computation. This note primarily describes the error sources in the numerical computation assuming the on-board estimates are exact.

2. The following are summaries of the numerical accuracy of the computation:
 - a) The display is completely invalid at ranges less than .12 nm.
 - b) The errors in the computation are more significant in earth orbit than lunar orbit.
 - c) The accuracy of the range computation is a function of inaccuracies of the integration routine and is independent of range with estimated maximum error of $\pm .1$ nm for earth orbit and $\pm .01$ nm for lunar orbit.
 - d) The accuracy of the range rate, dependent on the relative velocity and the line of sight, is also affected by the accuracy of the numerical integration routine. The accuracy of the line of sight degenerates as range decreases. An estimate of the maximum error in range rate is:

$$\dot{\Delta P} \leq .3 + (V + .3) \frac{.1}{P} \quad \text{for earth orbit}$$

$$\dot{\Delta P} \leq .1 + (V + .1) \frac{.01}{P} \quad \text{for lunar orbit}$$

where P is range in nm and V is relative velocity magnitude in fps.

- e) The V83/85 routine is not a good measure of cislunar navigation performance. (i. e. on board navigated state versus MSFN state.)

3.0 Background of Problem

There are two primary causes for error in the DSKY displays of range and range rate. They are:

- a. Errors in the on board computer's estimate of both states.
- b. Errors in the fixed point calculations of the V83 routine independent of those state errors above.

Normally, the errors caused by (a) above are more significant than those of (b). We will begin, however, by discussing the numerical errors first.

3.1 Numerical Errors in V83 Calculations

There are two primary sources of numerical errors in the computation of range and range rate. They are:

- a. Errors introduced by inaccuracies of the numerical integration scheme used to determine routine and velocity of both vehicles.
- b. Errors on the computation of R , \dot{R} and θ , given the position and velocity at time T .

In order to discuss the errors in the numerical accuracy of integration of position and velocity, it is important to review the logic used by the V83 Routine.

1. In all cases the program first reads the clock and does precision integration of both states to the clock time and stores the resulting state vectors as base vectors from which all further integrations are done.
2. Then,
 - a) if average G is on this vehicle's state is taken from R_N , V_N and $PIPTIME$ and the other vehicle is integrated from the base vector to $PIPTIME$ using precision integration (ENCKE)
 - b) if average G is not on, both base state vectors are integrated using conic routine (KEPLER) to current time.

NOTE: In COLOSSUS 2 and LUMINARY, if the LM is on the surface, the landing site vector \bar{R}_{LS} is used (automatically by the integration routine) and the new position of the LM is computed using the planetary orientation subroutine. The velocity is computed as the inertial velocity of the moon at \bar{R}_{LS} .

The errors in the resulting states are different for each case as well as different in earth or lunar orbit. Estimates for the sizes of these errors are provided in Section 3.2 below.

3.2 Orbital Integration Errors in V83

Several test runs were made in both earth and lunar orbit to determine an estimate of the magnitude of the relative position and relative velocity errors introduced by numerical problems in the integration routine. The numbers selected for purposes of this analysis were typical max values selected from the random values. As described in Section 3.1, two integrations are performed: one for the base vectors and then from those base vectors forward to time T. Because of these two integrations the typical max error was doubled for the worst case analysis for range and range rate:

200 meters, .1 m/sec for earth case

20 meters, .04 m/sec for lunar cases

Note that the errors are significantly smaller in lunar orbit.

3.3 Errors in Computation of Range, Range Rate

The errors introduced by the coding and fixed point arithmetic are insignificant compared to the orbital integration errors. However, because of the structure of the code there is presently one problem in the use of the ABVAL instruction such that if the scaled range is less than 2^{-21} the ABVAL results in zero range.

Since the position vectors are scaled at 2^{29} , the range is computed as zero arbitrarily at 256 meters because of the ABVAL problem. Also when the range is in this area the UNIT operation results in overflow and the unit line vector of sight becomes arbitrarily set at (1, 0, 0). This means the range rate is arbitrary. Therefore this display is meaningless at ranges of 256 meters or less. Another way of stating this is when the DSKY range is zero the entire display is invalid except for θ .

4.0 Errors Analysis of Range and Range Rate

The following establish notation and definitions:

$$P = \text{Range}$$

$$\dot{P} = \text{Range Rate}$$

$$\bar{R}_L, \bar{V}_L = \text{LM State}$$

$$\bar{R}_C, \bar{V}_C = \text{CM State}$$

$$\bar{R} = \text{Relative position vector or line of sight vector} = \bar{R}_L - \bar{R}_C$$

$$\bar{V} = \text{Relative Velocity} = \bar{V}_L - \bar{V}_C$$

$$\bar{u} = \text{unit line of sight} = \text{unit } \bar{R}$$

The following are equations for range and range rate

$$P = |\bar{R}_L - \bar{R}_C| = |\bar{R}|$$

$$\dot{P} = (\bar{V}_L - \bar{V}_C) \cdot \bar{u} = \bar{V} \cdot \bar{u}$$

The errors in P and \dot{P} given by ΔP and $\Delta \dot{P}$ as a function of the errors in the states $\Delta \bar{R}_L, \Delta \bar{R}_C, \Delta \bar{V}_L, \Delta \bar{V}_C$ is provided in Appendix A.

4.1 Error in Range

Appendix A shows by direct partial differentiation

$$\Delta P = \Delta \bar{R} \cdot \bar{u}$$

$$\Delta P \leq |\Delta \bar{R}|$$

We note that the error in range is the projection of the relative position error along the line of sight and that the error in range is not a function of range.

If we consider the error introduced by numerical accuracy of orbital integration (i. e. the on board estimates of the states are perfect) then the typical max error which is the same at any range is:

$$0 \lesssim \Delta P \lesssim .1 \text{ nm (200 m) for earth cases}$$

$$0 \lesssim \Delta P \lesssim .01 \text{ nm (20 m) for lunar cases}$$

For true error in the DSKY range we must add the relative velocity error along the line of sight due to poor on board estimates. This, of course, is variable and is dependent on trajectory, marking schedules, etc. The LM display of P is better than the CM because of the incorporation of range and range rate measurements in navigation.

4.2 Error in Range Rates

From Appendix A the error in range rate as a function of state errors:

$$\dot{\Delta P} = \Delta \bar{V} \cdot \bar{u} + \frac{1}{P} \bar{V}^T \left[I - \bar{u} \bar{u}^T \right] \Delta \bar{R}$$

The second term on the right hand side is difficult to work with for purposes of this analysis. Therefore, another way to arrive at $\dot{\Delta P}$ was selected:

$$\text{Let } \dot{P} = \bar{V} \cdot \bar{u}$$

Assume we have errors in \bar{V} and \bar{u} such that

$$\dot{P} + \Delta \dot{P} = (\bar{V} + \Delta \bar{V}) \cdot (\bar{u} + \Delta \bar{u}) \quad \text{NOTE: } \bar{u} + \Delta \bar{u} \text{ is a unit vector.}$$

Substituting \dot{P} from this equation we obtain:

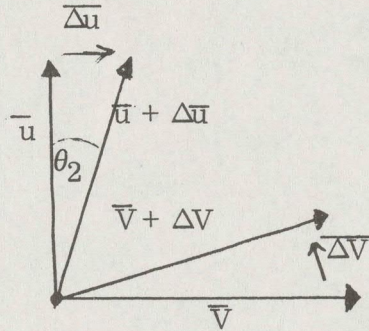
$$\Delta \dot{P} = \Delta \bar{V} \cdot \bar{u} + (\bar{V} + \Delta \bar{V}) \cdot \Delta \bar{u}$$

Examining this equation we have 2 parts:

Part one is $\Delta \bar{V} \cdot \bar{u}$ is the projection of the error in relative velocity along the line of sight and is maximum when ΔV is along the line of sight.

Part two, $(\bar{V} + \Delta \bar{V}) \cdot \Delta \bar{u}$ is the projection of the total relative velocity along the error in line of sight. In order to understand the physical significance of this observe the following diagram.

Figure 1



In this diagram the actual $\dot{P} = 0$, and the error in range rate is

$$\Delta \bar{V} \cdot \bar{u} + (\bar{V} + \bar{V}) \cdot \Delta \bar{u}.$$

or,

$$\begin{aligned} (\bar{u} + \Delta \bar{u}) \cdot \Delta \bar{V} &= |\bar{V} + \Delta \bar{V}| |\Delta \bar{u}| \cos \theta, \\ &\leq |\bar{V} + \Delta \bar{V}| |\Delta \bar{u}|. \end{aligned}$$

In addition if $|\bar{u} + \Delta \bar{u}| = 1$, and $|\bar{u}| = 1$; that is they are both unit vectors, then $|\Delta \bar{u}| \sim \sin \theta_2$ for where θ_2 is the angle between $\bar{u} + \Delta \bar{u}$ and \bar{u} and is assumed less than 15° . For this maximum case

$$(\bar{V} + \Delta \bar{V}) \cdot \Delta \bar{u} \leq (\bar{V} + \Delta \bar{V}) \sin \theta_2$$

Combining the worst case, we have

$$\boxed{\Delta \dot{P} \leq |\Delta \bar{V}| + |\bar{V} + \Delta \bar{V}| \sin \theta_2} \quad \text{where}$$

θ_2 is the angular error in the line of sight. We must examine the size of $\sin \theta_2$.

4.3 Angular Error in the Line of Sight

To examine $\sin \theta_2$, where θ_2 is the angle between the actual line of sight \bar{u} and the line of sight $\bar{u} + \Delta\bar{u}$ in error, we can approximate this by

$$\sin \theta_2 \approx \theta_2 \approx \frac{\Delta R}{P}$$

Assuming a constant value for ΔR such as maximum integration, error θ_2 increases as the range decreases.

For a worst case we assume the relative position error to be all perpendicular to the line of sight. The following is a table of values for $\sin \theta_2$.

Figure 2

Range (nm)	θ_2 EARTH $\Delta R = .1$ nm	θ_2 MOON $\Delta R = .01$ nm
100	.001	.0001
50	.002	.0002
25	.004	.0004
10	.01	.001
5	.02	.002
1	.1	.01
0.5	.2	.02
0.25	.4	.04

5.0 Using V83 for a check of Cislunar Navigation and other Nav Checks

5.1 Cislunar Navigation

It has been suggested that V83 be used during trans earth/lunar coast to determine the P and P between the MSFN state in the LM slots and on-board navigated state in the CM slots (at it was in C Prime Flight) as a measure of on-board navigation performance.

As we can see by the above discussion V83 is not accurate particularly in \dot{P} calculation at short ranges. Therefore, the values of range rate should not be utilized as any measure of cislunar navigation performance in correcting the velocity. The range, however, may be used as a measure of position error with consideration for integration errors ΔR ,

It is recommended that a special display of P and \bar{V} be provided where $\bar{V} = \bar{V}_L - \bar{V}_C$.

5.2 Other Nav Checks

Sometimes, if the same state vector is uplinked into both the LM and CM slots, V83 is used as a check to determine that these states are the same (i.e. P and $\dot{P} = 0$, θ is arbitrary).

Tests were run to determine the validity of this with the following results:

- a. If both states are identical and no navigation or burns are done the V83 display of P and \dot{P} will remain zero indefinitely.
- b. If P47 is used or any burn or navigation is done such that one of the states is changed then the display of P and \dot{P} will not necessarily be zero. One test run on COLOSSUS 249, for example, where the CM state was zapped using V66, then P47 turned on for 1200 secs showed the range growing from zero to .1 nm and the range rate from 0 to 1.2 fps. This, of course, is caused by the difference between average G integration and coasting flight precision integration. However, after P47 is terminated the two states are no longer the same and subsequent use of V83 would result in non-zero numbers.

APPENDIX A

ANALYSIS OF ΔP AND $\dot{\Delta P}$

1) RANGE

The range R is computed by the following:

$$\begin{aligned} P &= |R_L - R_C| \\ &= [(\bar{R}_L - \bar{R}_C) - (\bar{R}_L - \bar{R}_C)]^{\frac{1}{2}} \end{aligned}$$

where the error ΔP is computed by partial differentiation as follows:

$$\begin{aligned} \Delta P &= \frac{\delta P}{\delta \bar{R}_L} \cdot \Delta \bar{R}_L + \frac{\delta P}{\delta \bar{R}_C} \cdot \Delta \bar{R}_C \\ &= \frac{1}{2} [(\bar{R}_L - \bar{R}_C)]^{-\frac{1}{2}} [2(\bar{R}_L - \bar{R}_C)] \Delta \bar{R}_L \\ &\quad + \frac{1}{2} [(\bar{R}_L - \bar{R}_C) \cdot (\bar{R}_L - \bar{R}_C)]^{-\frac{1}{2}} [-2(\bar{R}_L - \bar{R}_C)] \Delta \bar{R}_C \\ &= \frac{(\bar{R}_L - \bar{R}_C)}{R} \cdot \Delta \bar{R}_L - \frac{(\bar{R}_L - \bar{R}_C)}{R} \cdot \Delta \bar{R}_C \end{aligned}$$

$$\Delta P = u \cdot \Delta \bar{R}$$

2) RANGE RATE

The range rate \dot{P} is computed as follows where \bar{V}_C and \bar{V}_L are the velocity of the CM and LM state

$$\dot{P} = (\bar{V}_L - \bar{V}_C) \cdot (\bar{R}_L - \bar{R}_C) / P$$

The error in range rate $\dot{\Delta R}$ would be expressed as follows:

$$\dot{\Delta P} = \frac{\dot{\delta P}}{\delta \bar{V}_L} \Delta \bar{V}_L + \frac{\dot{\delta P}}{\delta \bar{V}_C} \Delta \bar{V}_C + \frac{\dot{\delta P}}{\delta \bar{R}_L} \Delta \bar{R}_L + \frac{\dot{\delta P}}{\delta \bar{R}_C} \Delta \bar{R}_C$$

We compute each of the above partials as follows:

$$\frac{\dot{\delta P}}{\delta \bar{R}_L} = (\bar{V}_L - \bar{V}_C)^T \frac{\delta}{\delta \bar{R}_L} \left[(\bar{R}_L - \bar{R}_C)/P \right] + (\bar{R}_L - \bar{R}_C)/P \frac{\delta}{\delta \bar{R}_L} (\bar{V}_L - \bar{V}_C)$$

$$= (\bar{V}_L - \bar{V}_C)^T \left[\frac{P(I) - (\bar{R}_L - \bar{R}_C) \frac{\delta P}{\delta \bar{R}_L}}{P^2} \right]$$

$$= (\bar{V}_L - \bar{V}_C)^T \left[\frac{P(I) - (\bar{R}_L - \bar{R}_C) \cdot \text{UNIT}(\bar{R}_L - \bar{R}_C)^T}{P^2} \right]$$

$$= \frac{(\bar{V}_L - \bar{V}_C)^T}{P} \left[(I) - \bar{u} \cdot \bar{u}^T \right]$$

$$\frac{\dot{\delta P}}{\delta \bar{R}_C} = \frac{(\bar{V}_L - \bar{V}_C)}{R} \left[\bar{u} \cdot \bar{u}^T - I \right]$$

$$\frac{\dot{\delta P}}{\delta \bar{V}_L} = (\bar{R}_L - \bar{R}_C)/P$$

$$\frac{\dot{\delta P}}{\delta \bar{V}_C} = -(\bar{R}_L - \bar{R}_C)/P$$

∴ substituting the partials into equation for $\Delta \dot{P}$ we have

$$\begin{aligned}
 \Delta \dot{P} &= \frac{(\bar{R}_L - \bar{R}_C)}{P} \Delta \bar{V}_L - \frac{(\bar{R}_L - \bar{R}_C)}{P} \Delta \bar{V}_C \\
 &+ \frac{(\bar{V}_L - \bar{V}_C)}{P} \left[I - \bar{u} \cdot \bar{u}^T \right] \Delta \bar{R}_L \\
 &+ \frac{(\bar{V}_L - \bar{V}_C)}{P} \left[\bar{u} \cdot \bar{u}^T - I \right] \Delta \bar{R}_C \\
 &= (\Delta \bar{V}_L - \Delta \bar{V}_C) \cdot \bar{u} + \left[\frac{(\bar{V}_L - \bar{V}_C)}{P} \right]^T \left[I - \bar{u} \bar{u}^T \right] \Delta \bar{R}_L - \Delta \bar{R}_C \\
 \Delta \dot{P} &= \bar{\Delta V} \cdot \bar{u} + \frac{1}{P} \left[\bar{V}^T \left[I - \bar{u} \cdot \bar{u}^T \right] \bar{\Delta X} \right]
 \end{aligned}$$