# SPACE SHUTTLE GN & C EQUATION DOCUMENT

No. 4

(Revision 1)

Precision State and Filter Weighting Matrix Extrapolation

by

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M.I.T. Charles Stark Draper Laboratory

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for

National Aeronautics and Space Administration Systems Analysis Branch Guidance and Control Division Manned Spacecraft Center, Houston, Texas

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The publication of this report does not constitute approval by the National Aeronautics and Space Administration of the findings or the conclusions contained therein. It is published only for the exchange and stimulation of ideas.

#### **REVISION 1 CHANGES**

This document is a revised version of the previous one dated February 1971, and supersedes that document. The following is a summary of the major technical changes included in this revision:

1. The Filter-Weighting Matrix Extrapolation Equations have been modified and expanded to account for the cross-correlation terms when a weighting matrix involving two state vectors is being extrapolated. The concept of a (6 x d) filter-weighting sub-matrix corresponding to a single state vector has been introduced.

2. The earth's J<sub>2</sub> gravity field has been elevated to the status of the standard option for the perturbing acceleration.

### FOREWORD

This document is one of a series of candidates for inclusion in a future revision of MSC-04217, "Space Shuttle Guidance Navigation and Control Design Equations". The enclosed has been prepared under NAS9-10268, Task No. 15-A, "GN & C Flight Equation Specification Support", and applies to function 1 of the Orbital Navigation Module (ON2) as defined in MSC-03690 Rev. A, "Space Shuttle Orbiter Guidance, Navigation and Control Software Functional Requirements", dated 15 June 1971.

Gerald M. Levine, Director APOLLO Space Guidance Analysis Division

# TABLE OF CONTENTS

Section 1	Introduction
Section 2	Functional Flow Diagram
Section 3	Input and Output Variables
Section 4	Description of Equations
Section 5	Detailed Flow Diagrams
Section 6	Supplementary Information

Number of additional quantities, such as land mark locations or instrument biases, being

Constant describing dominant term of earth's oblateness

Special function of r and 8 defined in text

v

	Geocentric position vector at time 1
	Magnitude of geocentric position vector
	NOMENCI ATTIRE
	NOMENCLATORE
$\frac{a}{-d}(t)$	Perturbing acceleration at time t
c <sub>nom</sub>	Constant for adjustment of nominal step-size
d	Number of columns in the filter weighting sub- matrix W
E <sub>d</sub> (t)	Covariance matrix of dimension d
f(q)	Special function of q defined in text
G(t)	Gravity gradient matrix
<u>i</u> pole	Unit vector of earth's north polar axis expressed in reference coordinates
i	Unit vector in the direction of the position
1 TP	vector r had optionally w
I <sub>3</sub>	Three-dimensional identity matrix
j	Number of additional quantities, such as land-
	mark locations or instrument biases, being estimated.
	Reference comic velocity vector at time t
J 2	Constant describing dominant term of earth's oblateness
q	Special function of $\underline{r}$ and $\underline{\delta}$ defined in text

vi

$\frac{r}{0}$	Geocentric position vector at time $t_0$
<u>r</u> (t)	Geocentric position vector at time t
r(t)	Magnitude of geocentric position vector
$r_{con}(t)$	Reference conic position vector at time t
r <sub>con</sub> (t)	Magnitude of reference conic position vector at time t
r <sub>E</sub>	Mean equatorial radius of the earth
$\frac{r}{F}$ F	Geocentric position vector at time t $_{\rm F}$
rj	Intermediate values of $\underline{r}$
<sup>s</sup> pert	Switch indicating the perturbing accelerations to be included
sw	Switch controlling whether state or filter-
	weighting matrix integration is being per-
	formed (used only internally in routine)
ta	Initial time point
-0	Randers variable permanentian error in estimate
+	Time to which it is desired to extrapolate
٢	(r w) and optionally W
	(10, -0) and optionally $0$
15 V	Handom variante representing error in estimate
$\frac{\nabla}{2}$ 0	Geocentric velocity vector at time t <sub>0</sub>
<sup>v</sup> F	Geocentric velocity vector at time t $_{\rm F}$
$\frac{v}{con}(t)$	Reference conic velocity vector at time t
117	Filten weighting metnig of time t
<sup>vv</sup> 0	riter-weighting matrix at time to
max	Maximum value of   P   permitted (used as
W F	Filter-weighting matrix at time t F
₩_k,i	Inree-dimensional column vectors into which
	the filter-weighting matrix is partitioned
x	Independent variable in Kepler routine

Previous value of x x

- Vector random variable of dimension j representing  $\gamma(t)$ errors in the additionally estimated quantities such as landmark locations or instrument biases
- Position deviation vector of true position from  $\delta(t)$ reference conic position at time t

Maximum value of  $\delta$  permitted (used as rectificaδ max tion criterion)

- Δt Time-step size in numerical integration of differential equation
- Maximum permissible time-step size Δt max
- Nominal integration time-step size  $\Delta t_{nom}$

Time convergence tolerance criterion € t

- $\epsilon$ (t) Random variable representing error in estimate of position vector at time t
- $\eta(t)$ Random variable representing error in estimate of velocity vector at time t
- Earth's gravitational parameter μ
- v(t)Velocity deviation vector of true velocity from reference conic velocity at time t
- permitted (used as Maximum value of ν <sup>v</sup> max rectification criterion)

Time interval since last rectification Previous value of  $\tau$ 

TI

T

ø

Geocentric latitude

#### 1. INTRODUCTION

The Precision State and Filter Weighting Matrix Extrapolation Routine provides the capability to extrapolate any spacecraft geocentric state vector either backwards or forwards in time through a force field consisting of the earth's primary central-force gravitational attraction and a superimposed perturbing acceleration. The perturbing acceleration may be either the single dominant term (J<sub>2</sub>) of the earth's oblateness or a more complete expression involving all significant perturbation effects. The Routine also provides the capability of extrapolating the filter-weighting matrix along the precision trajectory. This matrix, also known as the "W-matrix", is a square root form of the error covariance matrix and contains statistical information relative to the accuracies of the state vectors and certain other optionally estimated quantities.

On any one call, the routine extrapolates only one state vector and only those six rows of the filter-weighting matrix relating to this state vector. Two calls are required to extrapolate two separate state vectors and a complete filter-weighting matrix pertaining to two state vectors. The complete extrapolated filter-weighting matrix is obtained by properly adjoining the two separately extrapolated submatrices of six rows each.

The routine is merely a coded algorithm for the <u>numerical</u> solution of modified forms of the basic differential equations which are satisfied by the geocentric state vector of the spacecraft's center of mass and by the filter-weighting matrix, namely:

$$\frac{d^2}{dt^2} \frac{r(t) + \frac{\mu}{r^3(t)} \frac{r(t)}{r(t)} = \underline{a}_d(t)$$

and

$$\frac{d}{dt}W(t) = F(t)W(t),$$

where  $\underline{a}_{d}(t)$  is the vector sum of all the desired perturbing accelerations, and F(t) is a matrix containing the gravity gradient matrix and the identity matrix in its off-diagonal sub-blocks.

Because of its high accuracy and its capability of extrapolaing the filter-weighting matrix, this routine serves as the computational foundation for precise space navigation. It suffers from a relatively slow computation speed in comparison with the Conic State Extrapolation Routine.

1-1

#### 2. FUNCTIONAL FLOW DIAGRAM

The Precision State and Filter Weighting Matrix Extrapolation Routine performs its functions by integrating modified forms of the basic differential equations at a sequence of points separated by intervals known as time-steps, which are not necessarily of the same size. The routine automatically determines the size to be taken at each step.

As shown in Fig. 1, the state vector and (optionally) the filter-weighting sub-matrix are updated one step at a time along the precision trajectory until the specified overall transfer time interval is exactly attained. (The size of the last time-step is adjusted as necessary to make this possible.)



Figure 1. Functional Flow Diagram Precision State and Filter Weighting Mairix Extrapolation Routine



Figure 1. Functional Flow Diagram Precision State and Filter Weighting Matrix Extrapolation Routine

#### 3. INPUT AND OUTPUT VARIABLES

The Precision State and Filter Weighting Matrix Extrapolation Routine has the following input and output variables;

#### Input Variables

 $(\underline{r}_0, \underline{v}_0)$ Geocentric state vector to be extrapolated

Time associated with  $(\underline{r}_0, \underline{v}_0)$  and  $W_0$ 

Time to which it is desired to extrapolate  $(\underline{r}_0, \underline{v}_0)$  and optionally  $W_0$ 

Filter-weighting sub-matrix to be extrapolated (optional) (W has dimension 6 x d)

Number of columns in filter-weighting submatrix (d = 0, 6, 7, ..., where 0 indicates no W-matrix extrapolation)

spert

dime prove

to

tF

Wo

Switch indicating the perturbing accelerations to be included. (spert = 1 implies J2 oblateness term only; s pert > 1 implies a more complete perturbing acceleration model (or models).)

#### **Output Variables**

 $(\underline{r}_{F}, \underline{v}_{F})$ 

Extrapolated geocentric state vector

Extrapolated filter-weighting sub-matrix of dimension 6 x d

WF

# 4. DESCRIPTION OF EQUATIONS

# 4.1 Precision State Extrapolation Equations

Since the perturbing acceleration is small compared with the central force field, direct numerical integration of the basic differential equations of motion of the spacecraft state vector is inefficient. Instead, a technique due to Encke is utilized in which only the deviations of the state from a reference conic orbit are numerically integrated. The positions and velocities along the reference conic are obtained from the Kepler routine.

At time  $t_0$  the position and velocity vectors,  $\underline{r}_0$  and  $\underline{v}_0$ , define an osculating conic orbit. Because of the perturbing accelerations, the true position and velocity vectors  $\underline{r}(t)$  and  $\underline{v}(t)$  will deviate as time progresses from the conic position and velocity vectors  $\underline{r}_{con}(t)$ and  $\underline{v}_{con}(t)$  which have been conically extrapolated from  $\underline{r}_0$  and  $\underline{v}_0$ . Let

$$\frac{\delta}{\nu}(t) = \underline{r}(t) - \underline{r}_{con}(t)$$

$$\frac{\nu}{\nu}(t) = \underline{v}(t) - \underline{v}_{con}(t)$$

be the vector deviations. It can be shown that the position deviation  $\delta$  (t) satisfies the differential equation

$$\frac{d^{2}}{dt^{2}} \underbrace{\frac{\delta}{\delta}(t) + \frac{\mu}{r_{con}^{3}(t)}} \left[ f(q) \underline{r}(t) + \underline{\delta}(t) \right] = \underline{a}_{d}(t)$$

with the initial conditions

$$\underline{\delta}(t_0) = \underline{0}, \ \underline{\nu}(t_0) = \underline{0}$$

where

$$q = \frac{(\underline{\delta} - 2\underline{r}) \cdot \underline{\delta}}{r^2}, \ f(q) = q \ \frac{3 + 3q + q^2}{1 + (1 + q)^{3/2}}$$

and  $\underline{a}_{d}(t)$  is the total perturbing acceleration. The above second order differential equation in the deviation vector  $\underline{\delta}(t)$  is numerically integrated by a method described in a later subsection.

The term

$$\frac{\mu}{r_{\rm con}^3} \left[ f(q) \underline{r}(t) + \underline{\delta}(t) \right]$$

must remain small, i.e. of the same order as  $\underline{a}_{d}(t)$ , if the method is to be efficient. As the deviation vector  $\underline{\delta}(t)$  grows in magnitude, this term will eventually increase in size. When

$$|\delta(t)| > 0.01 |r_{oon}(t)|$$
 or  $|\nu(t)| > 0.01 |v_{oon}(t)|$ 

or when

$$| \underline{\delta}(t) | > \delta_{\max} \text{ or } | \underline{\nu}(t) | > \nu_{\max}$$

a new osculating conic orbit is established based on the latest precision position and velocity vectors  $\underline{r}(t)$  and  $\underline{v}(t)$ , the deviations  $\underline{\delta}(t)$ and  $\underline{v}(t)$  are zeroed, and the numerical integration of  $\underline{\delta}(t)$  and  $\underline{v}(t)$ continues. The process of establishing a new conic orbit is called rectification.

The total perturbing acceleration  $a_d(t)$  is in general the vector sum of all the desired individual perturbing accelerations comprising the total force field, such as those due to the earth's oblateness, the gravitational attractions of the sun and moon, and the earth's atmospheric drag. Since many Shuttle applications will require only the perturbing effect of the dominant term  $J_2$  of the earth's oblateness, the use of only this term has been made a standard option in the routine diagrammed in Section 5. However, provision has been made for handling a completely general perturbing acceleration. The form of this perturbing acceleration will depend primarily upon the requirements of the Orbit Navigation function.

The explicit expression for the earth's  $\rm J_{2}$  oblateness acceleration alone is:

$$\underline{\mathbf{a}}_{\mathrm{d}} = -\frac{\mu}{r^{2}} \left\{ \frac{3}{2} J_{2} \left[ \frac{\mathbf{r}_{\mathrm{E}}}{r} \right]^{2} \left[ (1 - 5 \sin^{2} \phi) \underline{\mathbf{i}}_{\mathrm{r}} + 2 \sin \phi \underline{\mathbf{i}}_{\mathrm{pole}} \right] \right\}$$

where

 $\underline{i}_{r}$  is the unit position vector in reference coordinates,

i pole is the unit vector of the earth's north polar axis expressed in reference coordinates,

 $\sin \phi = \frac{i}{r} \cdot \frac{i}{pole}$ 

and

 $r_{\rm F}$  is the mean equatorial radius of the earth.

#### 4.2 Filter-Weighting (W) Matrix Extrapolation Equations

The position and velocity vectors which are maintained by the spacecraft's computer are only estimates of the actual values of these vectors. As part of the navigation technique it is also necessary for the computer to maintain statistical information about the position and velocity vectors. Furthermore, in particular applications it is necessary to include statistical data on various other quantities, such as landmark locations during Orbit Navigation and certain instrument biases during Co-orbiting Vehicle Navigation. The filter-weighting W-matrix is used for all these purposes.

If  $\underline{\epsilon}(t)$  and  $\underline{\eta}(t)$  are three dimensional vector random variables with zero mean which represent the errors in the estimates of a spacecraft's position and velocity at time t, then the six-dimensional state error covariance matrix  $E_{\underline{\epsilon}}(t)$  at time t is defined by:

$$E_{6}(t) = \begin{bmatrix} \frac{\underline{\epsilon}(t) \underline{\epsilon}(t)^{T}}{\underline{\eta}(t) \underline{\epsilon}(t)^{T}} & \frac{\underline{\epsilon}(t) \underline{\eta}(t)^{T}}{\underline{\eta}(t) \underline{\eta}(t)^{T}} \end{bmatrix},$$

where the bar represents the expected value or ensemble average at the fixed time t of each element of the matrix over which it appears. If  $\gamma(t)$  is a j-dimensional vector random variable with zero mean which represents the errors in the estimates of the j additionally estimated quantities such as landmark locations or instrument biases, then a (6+j) - dimensional state and other parameter covariance matrix  $E_{(6+i)}(t)$  is defined by:

$$E_{(6+j)}(t) = \begin{bmatrix} \frac{\varepsilon(t) \underline{\gamma}(t)^{T}}{\underline{\gamma}(t) \underline{\epsilon}(t)^{T}} & \frac{\underline{\eta}(t) \underline{\gamma}(t)^{T}}{\underline{\gamma}(t) \underline{\eta}(t)^{T}} \\ \underline{\gamma}(t) \underline{\epsilon}(t)^{T} & \underline{\gamma}(t) \underline{\eta}(t)^{T} & \underline{\gamma}(t) \underline{\gamma}(t)^{T} \end{bmatrix}$$

Further, if the statistical properties of the positions and velocities of two separate spacecraft are to be maintained, a twelvedimensional state covariance matrix is defined by:

of direct numer	T €P€P	$\frac{1}{\epsilon_{\rm P} \eta_{\rm P}}$	$\frac{\epsilon}{-P \epsilon T}$	<u>e</u> p <u>n</u> T
$\mathbf{E}_{\mathbf{r}}(\mathbf{t}) =$	$\frac{\eta}{P} \stackrel{\epsilon}{\epsilon} \stackrel{T}{P}$	$\underline{\eta}_{\mathrm{P}} \underline{\eta}_{\mathrm{P}}^{\mathrm{T}}$	$\frac{\eta}{\mathbb{P} \in \mathbf{T}}^{T}$	$\frac{\eta_{\rm P} \eta_{\rm T}}{1}$
12(1)	€T €P T	$\epsilon_{\rm T} \frac{\eta}{\eta} {\rm P}^{\rm T}$	$\epsilon_{\mathrm{T}} \epsilon_{\mathrm{T}}$	$\underline{\epsilon}_{\mathrm{T}} \underline{\eta}_{\mathrm{T}}^{\mathrm{T}}$
₫ <i>₹ ""(</i> 8+])'	$\frac{\eta}{\mathrm{T}} \frac{\epsilon}{\mathrm{P}}^{\mathrm{T}}$	$\frac{\eta}{T} \frac{\eta}{T} \frac{\eta}{P}$	$\frac{\eta}{T} \underbrace{\epsilon}_{T} \underbrace{T}^{T}$	$\frac{\eta}{T} T \frac{\eta}{T} T$

where the subscripts P and T refer to the primary and target vehicles, respectively.

And finally, if the statistical properties of the j additionally estimated quantities are also to be maintained along with the two state vectors, a (12 + j) state and other parameter covariance matrix  $E_{(12 + j)}(t)$  is defined by:

where  $I_{\beta}$  is the 3 x 3 identity matrix, the O's are zero matrices of the required dimensions, and the G(1) are the 3 x 3 conic gravity gradient matrices

$$E_{(12+j)}(t) = \begin{bmatrix} E_{12}(t) & \frac{\overline{e}_{P} \underline{\gamma}^{T}}{\underline{\eta}_{P} \underline{\gamma}^{T}} \\ \underline{\gamma} \underline{e}_{P}^{T} & \underline{\gamma} \underline{\eta}_{P}^{T} & \underline{\gamma} \underline{e}_{T}^{T} & \underline{\gamma} \underline{\eta}_{T}^{T} \\ \underline{\gamma} \underline{e}_{T}^{T} & \underline{\gamma} \underline{\gamma} \underline{\gamma}^{T} \end{bmatrix}$$

Rather than use one of the above covariance matrices in the navigation procedure, it is more convenient to use a matrix  $W_d(t)$  having the same dimension d as the covariance matrix  $E_d(t)$  and defined by:

$$E_{d}(t) = W_{d}(t) W_{d}(t)^{T}$$

The matrix  $W_d(t)$  is called the filter-weighting matrix, and is in a sense a square root of the covariance matrix.

Extrapolation of the  $W_d(t)$  matrix in time may be made by direct numerical integration of the differential equation which it satisfies: (where j = 0, 1, 2,... is the number of additionally estimated quantities )

$$\frac{d}{dt} W_{(6+j)}(t) = \begin{bmatrix} O & I_3 & | \\ G(t) & O & | & O_{(6xj)} \\ \hline & O_{(jx6)} & & | & O_{(jxj)} \end{bmatrix} W_{(6+j)}(t)$$

where I<sub>3</sub> is the  $3 \ge 3$  identity matrix, the O's are zero matrices of the required dimensions, and the G(t) are the  $3 \ge 3$  conic gravity gradient matrices

$$G(t) = \frac{\mu}{r^{5}(t)} \left[ 3 \underline{r}(t) \underline{r}(t)^{T} - r^{2}(t) I_{3} \right]$$

associated with the vehicle under consideration or with the primary (P) or target (T) vehicle.

Extrapolation of the  $W_d$  matrix may also be made by the following technique, which is somewhat simpler to implement in an on-board computer since matrix manipulations are reduced to more tractable vector manipulations.

If the d x d filter-weighting matrix  $W_d = \begin{bmatrix} w_{k,i} \end{bmatrix}$  is partitioned into three-dimensional column vectors  $\underline{w}_{k,i}$  which bear the subscripts of their first component:

$$w_{d} = \begin{bmatrix} \frac{w}{0}, 0 & \frac{w}{0}, 1 & \cdots & \frac{w}{0}, (d-1) \end{bmatrix}$$
  
$$\frac{w}{3}, 0 & \frac{w}{3}, 1 & \cdots & \frac{w}{3}, (d-1)$$

except for the last row where the  $\underline{w}_{k,i}$  vectors may be one or twodimensional if d is not divisible by three, then the previous first order differential equations are equivalent to:

$$\frac{d^{2}}{dt^{2}} \underline{w}_{0,i} = G(t) \underline{w}_{0,i}$$
with
$$\underline{w}_{3,i} = \frac{d}{dt} \underline{w}_{0,i}$$
i = 0, 1, ... (d - 1)
$$\underline{w}_{k,i} = \text{ constant for } k \ge 6$$

$$\frac{d^{2}}{dt^{2}} \underline{w}_{0,i} = G_{P}(t) \underline{w}_{0,i}$$

$$\frac{d^{2}}{dt^{2}} \underline{w}_{6,i} = G_{T}(t) \underline{w}_{6,i}$$
i = 0, 1, ..., (d - 1)
$$\frac{d^{2}}{dt^{2}} \underline{w}_{6,i} = G_{T}(t) \underline{w}_{6,i}$$
i = 0, 1, ..., (d - 1)
$$\underline{w}_{3,i} = \frac{d}{dt} \underline{w}_{0,i}$$

$$\underline{w}_{9,i} = \frac{d}{dt} \underline{w}_{6,i}$$

$$\underline{w}_{k,i} = \text{ constant for } k \ge 12$$

and

When written out in full, the above equations are:

$$\frac{d^{2}}{d t^{2}} \underline{w}_{0,i} = \frac{\mu}{\underline{r}_{p}^{5}(t)} \left\{ 3 \left[ \underline{r}(t) \cdot \underline{w}_{0,i}(t) \right] \underline{r}(t) - r^{2}(t) \underline{w}_{0,i}(t) \right\}$$
with
$$i = 0, 1, \dots, (d - 1)$$

$$\underline{w}_{3,i} = \frac{d}{d t} \underline{w}_{3,i}$$

$$\underline{w}_{k,i} = \text{ constant for } k \ge 6$$
and
$$\frac{d^{2}}{d t^{2}} \underline{w}_{0,i} = \frac{\mu}{r_{p}^{5}(t)} \left\{ 3 \left[ \underline{r}_{p}(t) \cdot \underline{w}_{0,i}(t) \right] \underline{r}_{p}(t) - r_{p}^{2}(t) \underline{w}_{0,i}(t) \right]$$

$$\frac{d^{2}}{d t^{2}} \underline{w}_{6,i} = \frac{\mu}{r_{T}^{5}(t)} \left\{ 3 \left[ \underline{r}_{T}(t) \cdot \underline{w}_{6,i}(t) \right] \underline{r}_{T}(t) - r_{T}^{2}(t) \underline{w}_{6,i}(t) \right\}$$
with
$$i = 0, 1, \dots, (d - 1)$$

$$\underline{w}_{3,i} = \frac{d}{d t} \underline{w}_{0,i}$$

$$\underline{w}_{9,i} = \frac{d}{d t} \underline{w}_{6,i}$$

$$\underline{w}_{k,i} = \text{ constant for } k \ge 12$$

These second-order differential equations may be integrated using the same numerical integration technique as is used for the spacecraft position vector. The vectors  $\underline{w}_{3,i}$  and  $\underline{w}_{9,i}$  bear the same relationship to the spacecraft velocity vector as the vectors  $\underline{w}_{0,i}$  and  $\underline{w}_{6,i}$  bear to the spacecraft position vector, and  $\underline{w}_{3,i}$  and  $\underline{w}_{9,i}$  are a by-product of the numerical integration of  $\underline{w}_{0,i}$  and  $\underline{w}_{6,i}$  just as the velocity vector is a by-product of the numerical integration of the nu

4-7

#### 4.3 Numerical Integration Method

The extrapolation of inertial state vectors and filter weighting matrices requires the numerical solution of two second-order vector differential equations, which are special cases of the general form

$$\frac{d^2}{dt^2} \underline{y}(t) = \underline{f}(t, \underline{y}(t), \underline{z}(t))$$

where

$$\underline{z} = \frac{d}{dt} \underline{y} .$$

Nystrom's standard fourth-order method is utilized to numerically solve this equation. The algorithm for this method is:

$$\begin{split} \underline{y}_{n+1} &= \underline{y}_n + \underline{z}_n \ \Delta t + \frac{1}{6} (\underline{k}_1 + \underline{k}_2 + \underline{k}_3) (\Delta t)^2 \\ \underline{z}_{n+1} &= \underline{z}_n + \frac{1}{6} (\underline{k}_1 + 2\underline{k}_2 + 2\underline{k}_3 + \underline{k}_4) \ \Delta t \\ \underline{k}_1 &= \underline{f}_n (t_n, \underline{y}_n, \underline{z}_n) \\ \underline{k}_2 &= \underline{f}_n (t_n + \frac{1}{2} \ \Delta t, \ \underline{y}_n + \frac{1}{2} \ \underline{z}_n \ \Delta t + \frac{1}{8} \ \underline{k}_1 (\Delta t)^2, \ \underline{z}_n + \frac{1}{2} \ \underline{k}_1 \ \Delta t) \\ \underline{k}_3 &= \underline{f}_n (t_n + \frac{1}{2} \ \Delta t, \ \underline{y}_n + \frac{1}{2} \ \underline{z}_n \ \Delta t + \frac{1}{8} \ \underline{k}_1 (\Delta t)^2, \ \underline{z}_n + \frac{1}{2} \ \underline{k}_2 \ \Delta t) \\ \underline{k}_4 &= \underline{f}_n (t_n + \Delta t, \ \underline{y}_n + \underline{z}_n \ \Delta t + \frac{1}{2} \ \underline{k}_3 (\Delta t)^2, \ \underline{z}_n + \underline{k}_3 \ \Delta t) \end{split}$$

where

$$\underline{y}_{n} = \underline{y}(t_{n}), \underline{z}_{n} = \underline{z}(t_{n})$$

and

 $t_{n+1} = t_n + \Delta t$ 

As can be seen, the method requires four evaluations of  $\underline{f}(t, \underline{y}, \underline{z})$  per integration step  $\Delta t$  as does the classical fourth-order Runge-Kutta method when it is extended to second-order equations. However, if  $\underline{f}$  is independent of  $\underline{z}$ , then Nystrom's method above only requires three evaluations per step since  $\underline{k}_3 = \underline{k}_2$ . (Runge-Kutta's method will still require four).

The integration time step  $\Delta t$  may be varied from step to step. The nominal integration step size is

$$\Delta t_{nom} = c_{nom} r_{con}^{3/2} / \sqrt{\mu}$$

where  $c_{nom}$  is a program constant. (The value  $c_{nom} = 0.3$  is recommended and implies that about 21 steps will be taken per trajectory revolution). The actual step-size is however limited to a maximum of  $\Delta t_{max}$ , which is also a program constant. (A value of about 4000 seconds is suggested.) Also, in the last step, the actual step size is taken to be the interval between the end of the previous step and the desired integration endpoint, so that the extrapolated values of the state or W-matrix are immediately available. Thus the integration step-size  $\Delta t$  is given by the formula

 $\Delta t = \pm \text{minimum} (|t_F - t|, \Delta t_{\text{nom}}, \Delta t_{\text{max}})$ 

where  $t_F$  is the desired integration end-point and t is the time at the end of the previous step. The plus sign is used if forward extrapolation is being performed, while the negative sign is used in the back-dating case.

# 5. DETAILED FLOW DIAGRAMS

This section contains detailed flow diagrams of the Precision State and Filter Weighting Matrix Extrapolation Routine.

Each input and output variable in the routine and subroutine call statements can be followed by a symbol in brackets. This symbol identifies the notation for the corresponding variable in the detailed description and flow diagrams of the called routine. When identical notation is used, the bracketed symbol is omitted.







Figure 2b. Detailed Flow Diagram



(Figure 2e)

Figure 2c. Detailed Flow Diagram







Figure 2e. Detailed Flow Diagram

#### 6. SUPPLEMENTARY INFORMATION

Encke's technique is a classical method in astrodynamics and is described in all standard texts, for example Battin (1964). The f(q) function used in Encke's technique (and in the lunar-solar perturbing acceleration computations) has generally been evaluated by a power series expansion; the closed form expression given here was derived by Potter, and is described in Battin (1964).

The oblateness acceleration in terms of a general spherical harmonic expansion may be calculated in a variety of ways; three different recursive algorithms are given in Gulick (1970). For low order expansions, especially those involving mostly zonal terms, an explicit formulation is generally superior computation-time-wise, as only the non-zero terms enter into the calculation. The general expression for the zonal terms is given by Battin (1964), while Zeldin and Robertson (1970) give explicit analytic expressions for each of the tesseral terms up through fifth order; hence all combinations of terms may easily be included in the oblateness acceleration by consulting the formulations in these references.

A full discussion of the use of covariance matrices in space navigation is given in Battin (1964). Potter (1963) suggested the use of the W-matrix and developed several of its properties. It should be noted that strictly the gravity gradient matrix G(t) should also include the gradient of the perturbing acceleration; however, these terms are so small that they may be neglected for our purposes. The use of only the conic gravity gradient, however, does not imply the W-matrix is being extrapolated conically. (Conic extrapolation of the W-matrix can be performed by premultiplying the W-matrix by the conic state transition matrix, which can be expressed in closed form). Rather the W-matrix is here extrapolated along the precision (perturbed) trajectory, as can be seen from the detailed flow diagram of Section 5.

6-1

The Nystrom numerical integration technique was first conceived by Nystrom (1925), and is described in all standard texts on the numerical integration of ordinary differential equations, such as Henrici (1962). Parametric studies carried out by Robertson (1970) on the general fourth-order Runge-Kutta and Nystrom integration techniques indicate that the "classic" techniques are the best overall techniques for a variety of earth orbiting trajectories in the sense of minimizing the terminal position error for all the trajectories, although for any one trajectory a special technique can generally be found which decreases the position error after ten steps by one or two orders of magnitude for only that trajectory. The classical fourth-order Runge-Kutta and Nystrom techniques are approximately equally accurate, but the latter possesses the computational advantage of requiring one less perturbing acceleration evaluation per step when the perturbing acceleration is independent of the velocity. This fact has been taken into account in the detailed flow diagram of Section 5. in that the extra evaluation is performed only when the perturbing acceleration depends explicitly on the velocity. Some past Apollo experience has suggested that extra evaluation effect with drag is so small as to be negligible; further analysis will confirm or deny this for the Space Shuttle. In regard to step-size, the constants and the functional form of the nominal and maximum time-step expressions have been determined by Marscher (1965).

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6-2

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R-1

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