**Modular Electronics Learning (ModEL) Project**

* SPICE ekt
  v1 1 0 dc 12
  v2 2 1 dc 15
  v3 2 3 4700
  v4 3 0 7100
  .dc v1 12 12 1
  .print dc v(2,3)
  .print dc i(v2)
  .end

**Phasor Mathematics**

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Chapter 1

Introduction

DC circuit calculations are fairly easy to perform because the quantities in question are unchanging over time. AC circuit calculations are much more complex by comparison because no voltage or current ever remains fixed at one value: all voltages and currents alternate by definition, generally following the pattern of a sinusoid. Another complicating factor is that often we encounter AC voltage and current “waves” that are out-of-step (i.e. phase-shifted) from each other, which means they add and subtract in non-intuitive ways.

A very useful tool for the simplification of AC circuit analysis is the use of complex numbers to represent voltages ($V$), currents ($I$), and impedances ($Z$). Any such quantity represented as a complex number is called a phasor because that complex-number quantity represents both the magnitude as well as the phase angle of that parameter. If we represent all AC circuit quantities as phasors, we find that many of the principles and laws familiar from our study of DC circuit analysis such as Ohm’s and Kirchhoff’s Laws still hold true. The only challenge remaining to AC circuit analysis once we embrace the notion of using phasors is the task of performing arithmetic operations such as addition, subtraction, multiplication, and division on complex numbers rather than the more customary “real” numbers used in DC circuit analysis.

Fortunately, many modern electronic calculators provide this arithmetic capability, allowing us to compute complex numbers almost as easily as we may compute real numbers. Unfortunately, many beginning electricity and electronics textbooks fail to show students how to use these amazing tools, instead forcing them to perform these calculations using trigonometric functions (which still require the use of an electronic calculator, incidentally, to compute with any speed).
Imagine a scenario where students are taught the concept of square roots for the very first time in their lives, as a mathematical tool useful for determining geometric lengths (e.g. the side-length of a square of given area, or the length of a right triangle’s hypotenuse). However, after being shown the utility of this new tool, those same students are then subject to the arbitrary requirement of performing every square-root computation using only the basic operations of addition, subtraction, multiplication, and division\(^1\). No student is allowed to use an electronic calculator with a square-root function, because that would be considered “cheating” by the standards of this curriculum.

Surely such a scenario would be considered odd, if not outright discouraging. Why handicap students in their computation, unless some higher educational purpose be fulfilled? I have yet to meet a mathematics instructor who forbids the use of electronic calculators to compute square roots. No higher purpose is served in manually computing roots, and by using the calculator as a procedural “amplifier” the student is relieved of an unnecessary burden, freed to devote mental effort to the task at hand.

I feel this same way about the use of phasor arithmetic to perform alternating current (AC) circuit computations. Phasors, like square roots, are a very useful mathematical tool for determining the relationships between voltage and current in AC circuits because they allow nearly all DC circuit principles and rules to be applied unchanged\(^2\) to AC circuits. However, performing phasor calculations “by hand” (i.e. treating every phasor like the side of a triangle, using trigonometric functions such as sine, cosine, and tangent to compute those phasors’ magnitudes and angles) is tedious work indeed. In fact, these computations are often so tedious that students lose sight of the original circuit problem while struggling to compute the answers. The result is a learning experience reduced to pages of calculations, while conceptual understanding gets displaced.

Many modern electronic calculator models offer simple means of entering complex (phasor) number values and performing complex (phasor) arithmetic, similar to how all “scientific” calculators offer trigonometric functions and logarithmic functions at the push of a single button. If students are taught how to use these phasor arithmetic features, they will find the transition from DC circuit analysis to AC circuit far more manageable. Furthermore, having dispensed with the tedium of “manual” phasor calculations, students’ minds are free to focus on the actual circuit quantities without being bogged down by pages of trigonometric computations.

For those instructors who insist students learn to perform all phasor calculations “the old-fashioned way” using trigonometric and basic arithmetic functions, I recommend requiring students do this only after they’ve mastered AC circuit calculations using a calculator to handle all the phasor arithmetic. Teach the essential circuit concepts first, relying on calculators to reduce tedium, and only then require students to do computations the hard way. If necessary, enforce a “simple calculators only” rule on assessments to ensure they cannot use the “easy” complex-number functions and must apply the more tedious methods you require.

The advice I am giving here is tested and true. This is how I have taught AC circuit analysis — first allowing the use of calculators capable of doing complex-number arithmetic, and later requiring students to do those same calculations “longhand” — and it yields fast results. Don’t hobble your

---

\(^1\)There are “manual” procedures for computing square roots using only the basic arithmetic operations, but it is extremely tedious.

\(^2\)Ohm’s Law, Kirchhoff’s Laws, network theorems, and the like may be applied successfully to AC circuits so long as all quantities are represented in complex-number form rather than as regular “real” numbers! This unifying principle is often omitted from AC electricity curricula, leaving the hapless student to conclude AC circuit calculations are wildly different than DC circuit calculations.
students’ understanding of AC circuits by stubbornly insisting on the exclusive use of tedious arithmetic. Give them a powerful mathematical tool that frees their minds to focus on circuit concepts, and lets them see the similarities between DC and AC circuits, and only then (if necessary) require them to learn a more tedious method.
Chapter 2

Simplified Tutorial

Spinning a magnetized rotor past sets of stationary “stator” coils (also called stator windings) will produce alternating (AC) voltages resembling sine waves when displayed on an oscilloscope. This is the basis of all electromagnetic generators:

If multiple pairs of stator windings exist at different positions around the rotor’s circumference, the AC voltages generated by each winding pair will be shifted in phase from one another because the rotor’s poles can never align with all stator windings simultaneously: each stator winding reaches its “peak” voltage at a different rotor position. For a simple two-pole generator such as the one shown, the angle separating those winding positions will be the exact same phase-shift angle separating their AC voltage waveforms. In this example, that shift is 90°.
The sinusoidal\(^1\) shape of these voltage waveforms over time is no coincidence. Rather, it is a geometric fact proven by plotting the horizontal and vertical projections of a point moving around the circumference of a circle:

This geometry gives us ways to fully represent quantities within AC circuits. DC circuits pose little challenge in this way because the quantities are largely unchanging, but with AC circuits we have voltages and currents that continually vary over time, as well as other quantities such as impedance where the phase shift between voltage and current must be known. For example, a battery in a DC circuit may output a constant voltage of 6 Volts, but an AC voltage source with a 6 Volt “peak” rating continually cycles from zero Volts to +6 Volts to zero Volts to −6 Volts to zero Volts repeatedly. The math we use must be capable of expressing this reality.

A type of mathematical quantity called a complex number serves this purpose well, and may be written in two different forms: polar and rectangular. The “polar” form of a complex number relates to the circle portion in the previous diagram, while the “rectangular” form relates to the projections of that circle (the sine and cosine waves).

---

\(^1\)The term “sinusoid” refers to a shape resembling a sine wave.
The following example should make this clear. Here we have both the polar and rectangular representations of our dual-winding AC generator’s output voltage at the instant in time where its shaft position is 150°. One winding pair has been labeled “Real” and the other “Imaginary”:

This generator happens to output 240 Volts peak, which we can tell directly from the magnitude portion of polar notation: $240 \, \text{V} \angle 150^\circ$. At that particular moment in time when the shaft reaches the 150° position, the two winding pairs are outputting $-207.85$ Volts and $+120$ Volts, respectively. This is directly represented by the rectangular notation: $-207.85 + j120$ V. The lower-case letter $j$ signifies an “imaginary” mathematical quantity, to distinguish one winding’s voltage from the other.

These two mathematical forms, $240 \, \text{V} \angle 150^\circ$ and $-207.85 + j120$ V, are just alternative ways of expressing the same thing: the voltage output by this generator at a specific moment in time. Even if our hypothetical generator only had one winding pair (making the other pair truly imaginary), there would still be merit in rectangular notation because the combination of real and imaginary terms completely define that point along the whole shape of the wave. Neither the real ($-207.85$ V) nor the imaginary ($+120$ V) terms by themselves uniquely identify the coordinate at 150°, but the combination of the two terms does.

---

Exercising the projected sine and cosine waves, we see there are two different points in time where the “real” voltage reaches $-207.85$ Volts and two different points in time where the “imaginary” voltage reaches $+120$ Volts, but these two conditions only exist simultaneously at one point in time (corresponding to 150°).
In fact, complex-number notation is useful for representing a great many quantities in AC circuits, not just generator voltage. We may represent any current, any voltage, and any impedance (the AC equivalent of resistance) in an AC circuit\textsuperscript{3} using complex numbers. If we do so, we find that nearly all the laws and properties learned for the analysis of DC circuits applies to AC circuits as well! In other words, the reason we learn how to use complex numbers is so we can make AC circuit analysis as simple as DC circuit analysis.

This use of complex numbers to represent and compute AC circuit quantities is called \textit{phasor} mathematics. The name comes from the fact that the complex number represents not just the magnitude of the quantity but also its \textit{phase relationship} to other quantities in the same circuit.

Polar notation is by far the most common complex-number notational form for representing real circuit quantities, because it directly relates to an oscilloscope’s view of an AC voltage or current: the peak value of the whole wave in addition to the angle at some specific moment in time. Rectangular notation exists as an aid to addition and subtraction, because these two arithmetic operations happen to be easier to perform in rectangular form than polar form.

 Thankfully, modern scientific calculators render these distinctions academic by providing convenient entry, display, and computation of complex numbers. This functionality makes complex-number arithmetic as easy as computing square roots, logarithms, trigonometric functions, or any other mathematical function commonly taken for granted. There are still some applications where you may need to switch from polar notation to rectangular in order to solve a particular problem, but this is a rare requirement (and the calculator will even perform this conversion for you!).

For example, consider the application of phasor arithmetic to the case of two AC voltage sources connected in series. The lower source outputs 4 Volts between points B and C, and serves as our “phase reference” with its angle arbitrarily set to 0°. The upper source outputs 3 Volts between points A and B, and has a phase shift that is +90° ahead of (leading) the 4 Volt source:

\[
\begin{align*}
V_{AB} &= 3 \text{ V} \angle 90° \\
V_{BC} &= 4 \text{ V} \angle 0° \\
V_{AC} &= 5 \text{ V} \angle 36.87°
\end{align*}
\]

Kirchhoff’s Voltage Law (KVL) applies to AC circuits just as well as to DC circuits, and therefore these two source voltages must \textit{add} to create a total voltage between points A and C. The only difference between AC and DC voltages is that we must use complex numbers (phasors) to do the addition. The right-hand photograph shows the lower source’s voltage value of 4 Volts \(\angle 0°\) added to the upper source’s voltage value of 3 Volts \(\angle 90°\) using a Texas Instruments model TI-36X Pro hand calculator with its display mode set to complex-polar form. The calculator automatically performs all the trigonometric operations necessary to execute this complex-number addition.

\textsuperscript{3}A few caveats apply, including the assumption that all voltages and currents share the same frequency.
Chapter 3

Full Tutorial

*Alternating current*, or *AC*, is the designation for any electrical quantity varying periodically over time. Even though “current” is part of this phrase, it is often used to describe *voltage* as well. The tempo at which an AC quantity varies is called the *frequency*, mathematically symbolized by the variable \( f \) and measured in the unit of *cycles per second*\(^1\) or *Hertz* (Hz). The magnitude of an AC quantity may be expressed in terms of the oscillation’s *peak* value, its *peak-to-peak* value, or its equivalent DC value based on its ability to perform physical work (called the *Root-Mean-Square* or *RMS* value). The following illustration shows two complete “cycles” of a sine wave, with the vertical axis representing the magnitude of the AC quantity and the horizontal axis representing time:

![Graph of a sine wave](image)

An AC voltage, for example, would possess a certain peak, peak-to-peak, and/or RMS voltage value as well as a period (measured in seconds) and a frequency (measured in cycles per second or Hertz). A greater magnitude appears “taller” on the vertical axis of a graph and a greater frequency appears “compressed” on the horizontal axis of a graph (i.e. more cycles within the same amount of time, or a shorter period).

A great many AC quantities are *sinusoidal* in shape; that is to say, their plot over time resembles that of a sine function or a cosine function over time. We must therefore begin our mathematical exploration of AC by reviewing some fundamental principles of trigonometry.

---

\(^1\)Since a “cycle” is technically not a unit of measurement like the second, frequency is sometimes expressed in units of *inverse seconds*, or s\(^{-1}\).
All trigonometric functions (e.g. sine, cosine, tangent) are based on the vertical and horizontal projections of a radius swept around a circle of constant diameter\(^2\). For this reason, one cycle of a sine-wave function represents one complete rotation around the circle.

The angle of rotation is commonly expressed by mathematicians in \textit{radians}: one radian being that angle associated with a section of the circle’s circumference equal to the radius length. This is why \(2\pi\) radians comprise one full rotation as shown in the illustration: because a full circle’s circumference is \(2\pi\) times longer than its radius. In engineering, the angular unit of the \textit{degree} is more commonly used to express angles of rotation, there being 360 degrees in a full circle (and therefore 360 degrees for one complete cycle of a sine or cosine wave).

The frequency of a wave is simply an expression of rotational speed, since each “cycle” of the wave is one rotation around the defining circle. The unit of Hertz (cycles per second) therefore is equivalent to \textit{revolutions per second}. Alternatively, frequency is sometimes referred to in terms of \textit{radians per second}, with 1 Hertz being equivalent to \(2\pi\) radians per second. When radians per second are the units of measurement, frequency is sometimes called \textit{angular velocity} and represented by the lower-case Greek letter Omega (\(\omega\)). Therefore, \(\omega = 2\pi f\).

\footnote{Trigonometry is commonly associated with right triangles, which seem at first to have no relation to a circle. However, if one considers the radius of that circle drawn to a point on the circumference to be the hypotenuse of a right triangle inscribed within the circle, that triangle’s opposite and adjacent side lengths represent the point’s vertical and horizontal positions.}
The analysis of DC circuits is possible by application of fundamental principles, such as the Conservation of Energy, the Conservation of Electric Charge, Ohm’s Law, Joule’s Law, and Kirchhoff’s Voltage and Current Laws. AC circuits are no different in this respect because the fundamental elements of electric charge, voltage, and charge motion (current) remain the same. What is different, however, is the quantification of variables such as voltage and current existing in a continual state of change, as well as the arithmetic necessary to add, subtract, multiply, and divide these ever-changing quantities. Ohm’s and Joule’s Laws require multiplication and division; Kirchhoff’s Laws require addition and subtraction. When quantities such as voltage, current, and resistance are constant these arithmetic operations are simple; when those same quantities continually vary these operations become considerably more complicated. Consider this graphical comparison of two quantities: a “DC” quantity with a constant value of 3, and an “AC” quantity being a sine wave with a peak value of 2:

$$\text{DC} = 3$$

$$\text{AC} = 2 \sin(x)$$

Since the majority of AC circuit variables are sinusoidal in nature, we will focus on the mathematical treatment of sine waves, beginning with addition and subtraction. Although AC circuits with non-sinusoidal voltages and currents exist, it is mathematically possible to express any repetitive oscillation of any shape as a summation of multiple sine and cosine waves, which means any mathematical techniques capable of handling sinusoidal functions may be extended to handle non-sinusoidal functions as well.

---

3The AC electricity created by an electromechanical generator oscillates in a sinusoidal pattern due to the circular motion of the generator’s magnetized rotor. Thus, the AC voltage produced by a spinning generator is an expression of the same phenomenon of sine and cosine waves being projections of a spinning radius. In fact, if an AC generator is equipped with two sets of stationary windings exposed to the spinning rotor’s magnetic field, each winding set 90 degrees apart from the other, one winding will output a sine wave while the other outputs a cosine wave.

4The mathematical technique for expressing a non-sinusoidal function as a series of sinusoidal functions is called the Fourier Transform.
Let us consider a pair of DC voltage sources connected in series with each other, reviewing how we should compute the “total” voltage of the source pair. In one scenario the two sources are aiding each other, and in the other scenario they oppose each other:

In the left-hand network where the two sources aid each other, the total voltage measured at point A with reference to point C (e.g. a voltmeter’s red test lead on A and black test lead on C), or $V_{AC}$, should be +7 Volts. In the right-hand network where the two sources oppose each other, the total voltage measured at point A with reference to point C (e.g. a voltmeter’s red test lead on A and black test lead on C), or $V_{AC}$, should be +1 Volt. This is all in accordance with Kirchhoff’s Voltage Law.

We may represent both scenarios graphically, showing how the two DC voltage quantities either add or subtract to make the resultant voltage $V_{AC}$:

The sum of the 4 Volt line and the 3 Volt line is a 7 Volt line; the difference between the 4 Volt line and the 3 Volt line is a 1 Volt line.

---

5It is worth noting that “AC” is used here to identify test points, and has nothing to do with $AC$ as used to abbreviate the phrase Alternating Current.
Let us consider the same series-connected voltages, but this time with those sources both being AC rather than DC. For the sake of illustration, we will consider source $V_{BC}$ to have a peak value of 4 Volts, and source $V_{AB}$ to have a peak value of 3 Volts, both at the same frequency. These sinusoidal plots were created using a computer program written in the C++ programming language, the computer evaluating the formulae $3 \sin x$ and $4 \sin x$ for values of $x$ ranging from 0 degrees to 360 degrees. The sum was similarly calculated via computer\(^6\), adding the two sine waves’ values together for every calculated value of $x$ ($\text{sum} = 3 \sin x + 4 \sin x$):

\[ V_{AB} + V_{BC} = 7 \text{ Volts peak} \]

\[ V_{AB} = 3 \text{ Volts peak} \]

\[ V_{BC} = 4 \text{ Volts peak} \]

Here we clearly see a sine wave with a peak value of 3 ($V_{AB}$) along with a sine wave with a peak value of 4 ($V_{BC}$), added together to make a third sine wave having a peak value of 7. This is rather intuitive: if we imagine these two sine waves adding together at every point in time, the sum must be zero at that exact angle when both waves cross zero, and the sum should peak at 7 Volts when each source is at its respective peak. Algebraic factoring proves this as well:

\[ V_{AB} = 3 \sin x \]

\[ V_{BC} = 4 \sin x \]

\[ V_{AB} + V_{BC} = 3 \sin x + 4 \sin x \]

\[ V_{AB} + V_{BC} = (3 + 4) \sin x \]

\[ V_{AB} + V_{BC} = 7 \sin x \]

\(^6\)There is no reason why this sum could not have been calculated by hand, using trigonometric tables or an electronic calculator to sum the respective values of two sine functions over a domain of 0 to 360 degrees. Computers merely expedite this laborious task.
If we were to program the computer to plot the sum of two AC voltages opposing each other (i.e. the two waveforms 180 degrees out of phase with each other), the result is not surprising either:

\[
\begin{align*}
V_{AB} + V_{BC} &= 1 \text{ Volt peak} \\
V_{BC} &= 4 \text{ Volts peak} \\
V_{AB} &= 3 \text{ Volts peak} \\
180^\circ \text{ phase shift}
\end{align*}
\]

Again, the computer is simply plotting two sine waves, \(4 \sin x\) and \(3 \sin(x + 180^\circ)\), and adding their respective values together at every computed value of \(x\) from 0 to 360 degrees to plot the third wave. The result is another sine wave with a peak value of 1. When \(x\) is equal to 0 degrees, 180 degrees, or 360 degrees, each of the two sine waves’ values is zero and therefore the summation of those waves is also zero at those points. When \(x\) is equal to 90 degrees or 270 degrees, each of the sine waves’ values reaches its peak (with opposite signs), resulting in a summation equal to \(-1\) or \(+1\).

It should be clear from these two graphical examples that the sum of two in-phase sine waves is analogous to the sum of two series-aiding DC voltages, while the sum of two sine waves precisely 180 degrees out of phase with each other is analogous to the sum of two series-opposing DC voltages. This much is simple, and appears no more complicated than any DC calculation. What is not clear yet, though, is what happens when we set the phase shift at some value other than 0 degrees or 180 degrees, because this condition has no DC analogue. In DC, voltage sources may (only) aid or oppose each other. To consider an AC example with a phase shift other than 0 degrees or 180 degrees is to consider a case where two voltage sources neither directly aid nor directly oppose each other.
Let us explore the summation of two sine waves phase-shifted from each other by 90 degrees. We will use the same peak amplitudes as before, 4 Volts peak and 3 Volts peak, programming the computer to add each of those waves’ values together at every computed value of $x$:

The result, according to the graph, is a sine wave with a peak value of 5 Volts and a phase shift that is approximately 40 degrees ahead of the 4 Volt (blue) wave. Those familiar with trigonometry should recognize the values 3, 4, and 5 as one of the unique sets of integer values that work as side-lengths for a right triangle:
This similarity between a 3-4-5 right triangle and the sum of sine waves (peak of 3 plus peak of 4 equaling peak of 5) is no coincidence. Recall that sine waves are really just the projection of a point's vertical position as it sweeps around the circumference of a circle. The sum of any two sine waves at the same frequency, therefore, must be the sum of those height-projections at every respective point around the circles.

The following illustrations show the two AC voltage sources’ sine waves as projections of circular radii. Each of the radii are called *phasors*, having respective lengths of 3 and 4. At the bottom is a summation of these phasors, forming a circle of radius 7 and a corresponding sine wave with a peak value of 7:

This graphically represents our first AC scenario, where a 3 Volt sine wave and a 4 Volt sine wave at the same frequency and in-phase with each other add together to make a 7 Volt sine wave.
Now let us analyze our second scenario by the same graphical method, projecting the sum of a 4 Volt sine wave and a 3 Volt sine wave that is 180 degrees out of phase:

Note how the phasors of the 3 Volt sine wave begin at 180 degrees and end at 270 degrees because that function is offset 180 degrees from the 4 Volt sine wave. The sum of these two sine waves, of course, is a sine wave having a peak value of only 1, since for every value of \( x \) the two phasors are pointing in opposite directions of each other and therefore subtract.
Finally, we will analyze our third scenario by showing the graphical sum of a 4 Volt sine wave and a 3 Volt sine wave that is 90 degrees out of phase:

Pay close attention to lower illustration, where the circle’s radius is defined by the position of the point at the end of two stacked phasors, one having a length of 4 and the other with a length of 3, offset from each other by 90 degrees. This circle has an effective radius of 5, because its radius is the hypotenuse of a 3-4-5 right triangle inscribed within the circle. Note also how the projected function exhibits its own unique phase shift, starting neither at zero like the $4 \sin x$ wave nor starting at the positive peak like the $3 \sin(x + 90^\circ)$ wave. The phase offset of this summation function is equal to the angle between the 4 and 5 sides of a 3-4-5 triangle (36.87 degrees), which means we may write it symbolically as $5 \sin(x + 36.87^\circ)$. Therefore, $3 \sin(x + 90^\circ) + 4 \sin x = 5 \sin(x + 36.87^\circ)$. 
Now that we have seen the graphical relationships between added sinusoid functions and how these apply to AC circuit quantities that add together (e.g. series-connected AC voltage sources), we must find some way to perform these calculations for the purpose of AC circuit analysis. Fortunately for us, many electronic calculators provide such functionality in the form of complex number arithmetic. A sinusoidal function having a peak value and a phase angle may be expressed in polar form as a complex number’s magnitude and angle. Using our previous example of two series-connected AC voltage sources at the same frequency, where one had a peak value of 4 Volts and the other had a peak value of 3 Volts (with a 90 degree shift), the two source values could be entered into a calculator as the polar-form complex numbers $4\angle0^\circ$ and $3\angle90^\circ$, respectively.

The following photograph shows this calculation performed on a Texas Instruments model *TI-36X Pro* hand calculator, with its display mode set to complex-polar form:

Calculations based on the treatment of electrical quantities as complex numbers are not limited to addition and subtraction. Complex numbers may be multiplied, divided, squared, square-rooted, etc. just like normal “real” numbers, which means we are able to perform calculations with AC circuit quantities just the same as with DC circuit quantities, using all the same laws and principles.

The fact that all these complex number operations are performed as easily as normal “real” numbers using a suitable hand calculator means we have a powerful tool for AC circuit analysis. If performing AC circuit calculations using Ohm’s Law and Kirchhoff’s Laws is your only goal, then there is nothing more you need to learn about this topic than where to obtain a suitable hand calculator capable of performing complex number arithmetic in polar form. The rest of this tutorial is dedicated to a deeper exploration of complex numbers and their relation to sinusoidal functions.

---

7 A “complex” number is a combination of a “real” quantity and an “imaginary” quantity. Complex numbers may be expressed in rectangular form where the real and imaginary portions are separately shown, or they may be expressed in polar form where the phasor’s length and angle are shown.

8 A notable exception is the calculation of power by multiplying voltage and current, which doesn’t work quite the same for AC as it does for DC. The calculation of power in AC circuits is a topic unto itself, covered in a different module.
Complex numbers and sinusoidal functions are related by a famous mathematical formula called Euler’s Relation, discovered by the Swiss mathematician Leonhard Euler (1707-1783):

\[ e^{j\theta} = \cos \theta + j \sin \theta \]

Where,
- \( e = \) Euler’s number (approximately equal to 2.718281828)
- \( \theta = \) Angle of phasor, in radians
- \( \cos \theta = \) Horizontal projection of a unit phasor (along a real number line) at angle \( \theta \)
- \( j = \) Imaginary “operator” equal to \( \sqrt{-1} \), alternatively represented as \( i \)
- \( j \sin \theta = \) Vertical projection of a unit phasor (along an imaginary number line) at angle \( \theta \)

To illustrate, we will apply Euler’s relation to a unit \(^9\) phasor having an angular displacement of \( \frac{3\pi}{8} \) radians (67.5 degrees):

\[ e^{j\frac{3\pi}{8}} = \cos \frac{3\pi}{8} + j \sin \frac{3\pi}{8} \]
\[ e^{j\frac{3\pi}{8}} = 0.3827 + j0.9239 \]

The sine wave exists as the vertical projection of the rotating phasor on the “imaginary” \((j\) or \(i\)) axis, while the cosine wave exists as the horizontal projection of that same phasor on the “real” axis. At the particular angle arbitrarily chosen for this example (\(3\pi/8\) radians) the vertical quantity happens to be \(j0.9239\) while the horizontal quantity happens to be 0.3827.

---

\(^9\)The term “unit phasor” simply refers to a phasor with a length of 1 (“unity”).
For “unit” circles having a radius value of 1, Euler’s Relation perfectly describes the projected cosine and sine functions for any angle $\theta$. If our goal, though, is to represent voltage and current quantities with peak values other than 1, we must append a multiplication factor to Euler’s Relation accounting for the amplitude\(^{10}\) ($A$) of the sinusoidal function:

$$Ae^{j\theta} = A\cos \theta + jA\sin \theta$$

Using this version of Euler’s Relation, we may perfectly describe the functions of a circle having a radius length of $A$:

---

\(^{10}\)The terms amplitude and magnitude are often synonymous. Here, we are using the latter to describe the intensity of the circuit quantity (e.g. voltage or current) at any given moment in time, and the former to describe only the peak value it reaches.
An important extension of Euler’s Relation to real-world AC circuit applications is the inclusion of time as one of the variables, since AC quantities oscillate over time. Recall that the angular velocity of a rotating phasor \( \omega \) is equivalent to its frequency measured in radians per second rather than Hertz. The angle of a phasor at any point in time, therefore, is the product of its angular velocity (radians/second) and the specific value of time (seconds):

\[
\theta = \omega t \quad \text{[radians]} = \text{[radians/second][seconds]}
\]

Therefore, we may substitute \( \omega t \) in place of \( \theta \) in Euler’s Relation to now describe phasors and their sinusoidal projections as functions of time:

\[
A e^{j\omega t} = A \cos(\omega t) + jA \sin(\omega t)
\]

This too has a graphical representation, with the real, imaginary, and time axes all perpendicular to each other, forming a three-dimensional graph:

The phasor’s tip traces a spiral path as it rotates around the circle and progresses along the time axis. When viewed from the end (with the time axis pointing away from us), all we see is the phasor’s rotation. When viewed from the side, we see either a cosine wave or a sine wave depending on whether we are looking at the phasor’s horizontal projection or its vertical projection. A physical model of this is either a compression-style coil spring or a corkscrew: viewed from the end you see a circle; viewed from the side you see a sinusoid.
This three-dimensional graph has an electrical analogue as well: an AC generator with a spinning magnetic rotor (representing the rotating phasor) and two sets of stationary coils (“windings”), one winding generating a sine wave and the other generating a cosine wave:

\[
A e^{j\theta} = A \cos \theta + jA \sin \theta
\]

For a generator with peak output voltage \( A \) and rotor shaft angle \( \theta \)

The angular offset between these two sets of stator windings accounts for the 90 degree (\( \frac{\pi}{2} \) radian) phase shift between the two sinusoidal output voltages. One winding pair experiences the spinning rotor’s magnetic field one-quarter rotation before the other winding pair. The two sets of windings essentially “view” the spinning rotor from different perspectives. Euler’s Relation predicts the voltage developed at each winding based on the maximum (peak) voltage generated by either winding \( (A) \) and the angular position of the rotor at that instant in time \( (\theta) \).

A surprising amount of electrical engineering is founded on Euler’s Relation. The equivalence between sinusoidal functions \( (\cos \theta + j \sin \theta) \) and exponential functions \( (e^{j\theta}) \) greatly simplifies a number of different mathematical analyses, and forms the basis for the rules we will use to add, subtract, multiply, and divide AC circuit quantities.
Although Euler’s Relation is the basis of AC circuit mathematics due to its ability to completely represent a phasor at any point in time, the full form of Euler’s Relation is often omitted for simplicity’s sake. In its place, electrical practitioners often use simplified mathematical notation to describe phasors. The two sides of Euler’s Relation – the exponential \((Ae^{j\theta})\) side and the trigonometric \((A\cos \theta + jA\sin \theta)\) side – are instead referred to as polar and rectangular, respectively. “Polar” refers to the description of a phasor on a polar plot: an arrow pointing away from the plot’s center (pole) with a specified length \(A\) and angle from horizontal \(\theta\). “Rectangular” refers to the description of a phasor’s tip on a rectangular plot: the “real” term \(A\cos \theta\) describing its position on the horizontal axis and the “imaginary” term \(jA\sin \theta\) describing its position on the vertical axis.

Shorthand phasor notation consists of the phasor’s amplitude followed by an angle symbol \((\angle)\) and the angle value for polar notation, and the real and imaginary coordinates for rectangular notation:

<table>
<thead>
<tr>
<th>Full notation</th>
<th>Shorthand notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Ae^{j\theta})</td>
<td>(A\angle\theta)</td>
</tr>
<tr>
<td>(A\cos \theta + jA\sin \theta)</td>
<td>(x + jy)</td>
</tr>
</tbody>
</table>

For illustrative purposes we will consider all these phasor representations at an amplitude \((A)\) of one and at four different angle values, one for each axis: 0 radians, \(\pi/2\) radians, \(\pi\) radians, and \(3\pi/2\) radians (corresponding to 0, 90, 180, and 270 degrees, respectively). All four examples are shown in written form as well as graphical (i.e. a phasor diagram):

<table>
<thead>
<tr>
<th>Angle ((\theta))</th>
<th>Exponential</th>
<th>Trigonometric</th>
<th>Rectangular</th>
<th>Polar</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 radians = 0°</td>
<td>(e^{j0})</td>
<td>(\cos 0 + j\sin 0)</td>
<td>(1 + j0 = 1)</td>
<td>(1 \angle 0^\circ)</td>
</tr>
<tr>
<td>(\pi/2) radians = 90°</td>
<td>(e^{j\pi/2})</td>
<td>(\cos 90^\circ + j\sin 90^\circ)</td>
<td>(0 + j1 = j)</td>
<td>(1 \angle 90^\circ)</td>
</tr>
<tr>
<td>(\pi) radians = 180°</td>
<td>(e^{j\pi})</td>
<td>(\cos 180^\circ + j\sin 180^\circ)</td>
<td>(-1 + j0 = -1)</td>
<td>(1 \angle 180^\circ)</td>
</tr>
<tr>
<td>(3\pi/2) radians = 270°</td>
<td>(e^{j3\pi/2})</td>
<td>(\cos 270^\circ + j\sin 270^\circ)</td>
<td>(0 - j1 = -j)</td>
<td>(1 \angle 270^\circ)</td>
</tr>
</tbody>
</table>

The third example – at an angle of \(\pi\) radians (180 degrees) – is uniquely elegant because it relates several important mathematical constants \((e, i, \pi, 1,\) and 0\) in a single formula. Here we will use the common mathematical symbol for an imaginary number \((i)\) instead of the symbol typically used by electrical practitioners \((j)\):

\[e^{i\pi} = -1 \quad \text{or} \quad e^{i\pi} + 1 = 0\]
The four basic arithmetic operations (addition, subtraction, multiplication, and division) are shown here in general form using two phasors, one with amplitude $A$ and angle $M$, and the other with amplitude $B$ and angle $N$:

$$Ae^{jM} + Be^{jN} = (A \cos M + B \cos N) + j(A \sin M + B \sin N)$$

$$Ae^{jM} - Be^{jN} = (A \cos M - B \cos N) + j(A \sin M - B \sin N)$$

$$Ae^{jM} \times Be^{jN} = AB e^{j(M+N)}$$

$$Ae^{jM} \div Be^{jN} = \frac{A}{B} [e^{j(M-N)}]$$

To give a more concrete example, we may perform all these arithmetic operations on the phasors $4 \angle 0^\circ$ and $3 \angle 90^\circ$, using shorthand notation:

$$4 \angle 0^\circ + 3 \angle 90^\circ = 4 \cos 0^\circ + 3 \cos 90^\circ + j(4 \sin 0^\circ + 3 \sin 90^\circ) = 4 + j3$$

$$4 \angle 0^\circ - 3 \angle 90^\circ = 4 \cos 0^\circ - 3 \cos 90^\circ + j(4 \sin 0^\circ - 3 \sin 90^\circ) = 4 - j3$$

$$4 \angle 0^\circ \times 3 \angle 90^\circ = 12 \angle 90^\circ$$

$$4 \angle 0^\circ \div 3 \angle 90^\circ = \frac{4}{3} \angle -90^\circ$$

Addition and subtraction in polar form necessitates the use of sine and cosine functions. If, however, the phasors in question are already cast in rectangular form, addition and subtraction is seen to be nothing more than combining the real and imaginary terms. Repeating the first two lines with $4 \angle 0^\circ$ expressed as $4 + j0$ and $3 \angle 90^\circ$ expressed as $0 + j3$:

$$(4 + j0) + (0 + j3) = (4 + 0) + (j0 + j3) = 4 + j3$$

$$(4 + j0) - (0 + j3) = (4 - 0) + (j0 - j3) = 4 - j3$$

It should be clear from these examples that addition and subtraction are best performed in rectangular form while multiplication and division lend themselves best to polar form.
If we ever find ourselves needing to convert from rectangular form into polar form, we may do so using standard trigonometric methods. All we need to do is regard the real and imaginary portions of the rectangular-form phasor as side-lengths of a right triangle inscribed within a circle of radius $A$ equal to the amplitude (peak magnitude) of the phasor in question:

![Diagram of polar form](image)

The polar angle $\theta$ may be calculated three different ways, based on side $y$ of the triangle being opposite of $\theta$ and side $x$ being adjacent to angle $\theta$, knowing that the tangent function is the ratio between opposite and adjacent, the cosine function is the ratio between adjacent and hypotenuse, and that the sine function is the ratio between opposite and hypotenuse:

$$
\theta = \tan^{-1}\left(\frac{y}{x}\right) \quad \theta = \cos^{-1}\left(\frac{x}{A}\right) \quad \theta = \sin^{-1}\left(\frac{y}{A}\right)
$$

The polar amplitude $A$ may be directly calculated from side lengths $x$ and $y$ using the Pythagorean Theorem:

$$
A = \sqrt{x^2 + y^2}
$$

---

11A misconception I’ve often encountered with students of trigonometry is that they assume the vertical side of a right triangle must always be the “opposite” and the horizontal side must always be the “adjacent”. This is not necessarily true, and in fact is only true when the angle in question is the one between the horizontal side and the hypotenuse. The terms “opposite” and “adjacent” refer to the particular angle being referenced. If we were to consider the angle between the $A$ and $y$ sides of our triangle, for example, then $x$ would be the side opposite that angle and $y$ would be the side adjacent to that angle.
Now that we have explored the arithmetic of phasors, we may return to our problem of calculating the total voltage from two series-connected AC voltage sources. The result has already been determined by graphical solution (a computer summing all respective points of $4\sin x$ and $3\sin(x + 90^\circ)$ to yield a sine wave having an amplitude of 5) as well as by hand calculator (the TI-36X Pro adding $4\angle 0$ and $3\angle 90^\circ$ to arrive at $5\angle 36.86989765$), and so we show it again in the illustration:

Let us calculate this same sum without the benefit of a computer plotting hundreds of points or a hand calculator capable of performing complex-number arithmetic. Since we know the electrical principle at work here is Kirchhoff’s Voltage Law, we know the total voltage $V_{AC}$ must be equal to the sum of the voltages $V_{BC}$ and $V_{AB}$. We also know that phasor sums are most easily computed in rectangular form, and so our first task is to convert the two voltage source values from polar form into rectangular form:

$$V_{BC} = 4\angle 0^\circ = 4\cos 0^\circ + j4\sin 0^\circ = 4 + j0$$

$$V_{AB} = 3\angle 90^\circ = 3\cos 90^\circ + j3\sin 90^\circ = 0 + j3$$

The total voltage of these two series-connected sources must therefore be:

$$V_{BC} + V_{AB} = (4 + j0) + (0 + j3) = 4 + j3$$

The result, $4 + j3$, while mathematically correct, does not relate directly to the indication given by a voltmeter connected between points A and C. Polar form would be a more realistic representation, giving us the amplitude (peak) value of the AC voltage as well as the phase shift angle. Therefore, our final step is to convert rectangular form into polar form:

$$\theta = \tan^{-1} \left( \frac{3}{4} \right) = 36.87^\circ$$

$$A = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5 \text{ Volts}$$

An AC voltage having an amplitude of 5 Volts and a phase angle of $36.87^\circ$ is our result, and therefore $V_{AC} = 5\angle 36.87^\circ$. 
The benefit of phasor mathematics is that it gives us a tool to apply fundamental laws and principles of DC circuits to AC circuits. This is incredibly important, as without such a tool we would have to approach AC circuits quite differently than DC circuits. As it is, all we need is a new way to perform the basic arithmetic operations of addition, subtraction, multiplication, and division incorporating amplitude and phase shift, and we are able to re-purpose all the concepts learned on DC circuits to AC circuits. It is the goal of scientific exploration to unify knowledge by seeking commonalities between seemingly disparate phenomena. Phasor mathematics is such a bridge, unifying DC and AC circuit analysis.

All mathematical techniques have practical limitations, though, and so we must outline some caveats of phasor arithmetic:

- The phasor arithmetic examples we’ve explored all assume a constant and equal frequency for all quantities. A 4 Volt source at 0 degrees adds to 3 Volt source at 90 degrees to make 5 Volts at 36.87 degrees, but only if those two voltage sources output precisely the same frequency. If the two quantities in question have differing frequencies, it means their phasors rotate around the circle at different speeds, and therefore their relative phase angles will not be constant.

- The phase angle specified for any AC waveform (i.e. voltage or current) is always relative to some arbitrary reference, and not absolute. For the 4 Volt and 3 Volt sources we keep using as an example, what we mean when we say the 4 Volt source has an angle of 0 degrees and the 3 Volt source has an angle of 90 degrees is that the 4 Volt source is our phase reference for the circuit. The 3 Volt source’s 90 degree angle simply means that source is 90 degrees ahead of the reference source at all times. We could have made the 3 Volt source our phase reference at 0 degrees and specified the 4 Volt source as having a phase angle of \(-90\) degrees, and the sum would still be 5 Volts.

- Multiplication or division of two sinusoids yields a result with a different frequency. This is what makes power calculations complicated in AC circuits. Joule’s Law \((P = IV)\) requires multiplying current by voltage, which means the frequency of power in an AC circuit is actually different than the frequency of the voltage or current! The amplitude of the calculated power will be accurate, but the phase angle of that product of voltage and current will not refer to a constant phase shift as it does between the voltage and current waveforms, and so will not have the same meaning. We will explore this topic in greater detail in another module.

- In developing our mathematical definition of a phasor – showing the rotating radius project as a sinusoidal waveform – the phasor’s length represents the peak amplitude of that quantity. For example, a voltage specified as \(3\angle90^\circ\) according to this definition has an amplitude of 3 Volts peak. In practice this definition is not strictly adhered to. In fact, it is quite common for electrical practitioners to specify the amplitude of all phasors in a circuit as Volts and Amperes \(RMS\) rather than peak. So long as every single phasor in that circuit is specified the same way, this is not a problem. Confusion results when some phasors are expressed in peak Volts or Amperes while others are assumed to express RMS Volts or Amperes.
Chapter 4

Derivations and Technical References

This chapter is where you will find mathematical derivations too detailed to include in the tutorial, and/or tables and other technical reference material.
4.1 Derivation of Euler’s Relation

Leonhard Euler was a Swiss mathematician who lived from 1707 to 1783, and was responsible for discovering what we now call Euler’s Relation, a mathematical formula relating exponential functions ($e^x$) to trigonometric functions ($\sin x$, $\cos x$). This relation forms the basis for phasor arithmetic, which is an incredibly useful tool for simplifying calculations in AC circuits where voltages and currents take the form of sinusoidal functions.

To begin, we must introduce the concept of a mathematical series, which is a sum of successive terms. The symbol used to denote a sum of many terms is the Greek capital letter “sigma” ($\Sigma$), with labels below and above specifying the beginning and end of the series. The following example is a form of “geometric” series of $x$ to the $n$ power, with $n$ ranging from zero to infinity:

$$\sum_{n=0}^{\infty} x^n$$

Expanded, it is equivalent to the following expression:

$$\sum_{n=0}^{\infty} x^n = x^0 + x^1 + x^2 + x^3 + x^4 + \cdots + x^\infty$$

Infinite series may be approximated using digital computers to perform the term calculations and summation, with the result approaching ever closer to the true value with every iteration.

Another important mathematical concept necessary to understand Euler’s Relation is that of the factorial. Symbolized by an exclamation mark, a “factorial” is the product (multiplication) of all whole numbers from the starting value down to one. Shown here are some factorial examples to illustrate:

- $0! = 1$ (by definition)
- $1! = 1 \times 1 = 1$
- $2! = 2 \times 1 = 2$
- $3! = 3 \times 2 \times 1 = 6$
- $4! = 4 \times 3 \times 2 \times 1 = 24$
- $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
- $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$
Euler’s Constant, symbolized by the letter $e$, happens to be the result of an infinite series of factorials:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

The partial sum (up to $n = 7$) appears as follows:

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!}$$

$$1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040}$$

It should be clear to see how each successive term becomes dramatically smaller than the one before it, which means the sum will asymptotically approach some final value. In this case, the final value happens to be $e$. If we examine a table of figures showing the summation of these terms from $n = 0$ to $n = 7$, the convergence becomes clear:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n!$</th>
<th>$\frac{1}{n!}$</th>
<th>$\Sigma \frac{1}{n!}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.5</td>
<td>2.5</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>0.16667</td>
<td>2.66667</td>
</tr>
<tr>
<td>4</td>
<td>24</td>
<td>0.04167</td>
<td>2.70833</td>
</tr>
<tr>
<td>5</td>
<td>120</td>
<td>0.00833</td>
<td>2.71667</td>
</tr>
<tr>
<td>6</td>
<td>720</td>
<td>0.00139</td>
<td>2.71806</td>
</tr>
<tr>
<td>7</td>
<td>5040</td>
<td>0.00020</td>
<td>2.71825</td>
</tr>
</tbody>
</table>

The value of $e$ is approximately 2.718281828459045, as we can see here how closely the series approaches this value after only seven iterations.

Exponential functions (i.e. powers of $e$) may also be calculated by infinite series, the basic exponential function $e^x$ being equivalent to the following:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots + \frac{x^\infty}{\infty!}$$
Certain trigonometric functions are also infinite series of factorials. For example, the cosine and sine functions (with angle \( x \) expressed in units of radians rather than degrees) are equivalent to the following series:

\[
\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}
\]

\[
\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}
\]

Shown here are partial sum expansions up to \( n = 5 \) for both cosine and sine functions:

\[
\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!}
\]

\[
\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}
\]

If we closely examine these two series and compare them with the exponential series (\( e^x \), shown below), some similarities should become apparent. Pay particular attention to the denominators of the fractional terms:

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots + \frac{x^n}{n!}
\]

Notice how the exponential series contains both odd- and even-numbered factorial terms, whereas the cosine series contains only even-numbered factorials and the sine series only odd-numbered factorials in their denominators. Note also how the cosine series begins with the term 1 while the sine series begins with \( x \), and how the exponential series contains both 1 and \( x \). Euler noticed these patterns too, and reasoned that the exponential series was some combination of the cosine and sine series. The major incongruency between the cosine, sine, and exponential series is the signs of the terms: the cosine and sine functions contain terms of alternating sign (i.e. positive, then negative, then positive, etc.) whereas the exponential series contains only positive terms.

Euler’s stroke of genius was in recognizing that imaginary numbers could be used to harmonize the signs of these terms, and create an equivalence between the exponential series and the cosine/sine series. Recall that an “imaginary” number is found by taking the square root of a negative number, with \( i \) or \( j \) used to express \( \sqrt{-1} \). If \( j = \sqrt{-1} \), then \( j^2 = -1 \) and \( j^3 = -j \) and \( j^4 = 1 \), etc.
4.1. DERIVATION OF EULER’S RELATION

Euler substituted \( jx \) for \( x \) in the exponential series, and made \( j \) a multiplying coefficient in the sine series, to show that the exponential series could be made the sum of the cosine and sine series. First, we will see what happens when we substitute \( jx \) for \( x \) in the exponential series:

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots
\]

\[
e^{jx} = 1 + jx + \frac{(jx)^2}{2!} + \frac{(jx)^3}{3!} + \frac{(jx)^4}{4!} + \frac{(jx)^5}{5!} + \frac{(jx)^6}{6!} + \frac{(jx)^7}{7!} + \cdots
\]

\[
e^x = 1 + jx - \frac{x^2}{2!} - j \frac{x^3}{3!} + \frac{x^4}{4!} + j \frac{x^5}{5!} - \frac{x^6}{6!} - j \frac{x^7}{7!} + \cdots
\]

Look closely at what happens to the \( j \) terms in the middle equation as they become raised to powers of 2, 3, 4, 5, and beyond. Recall that \( j^2 = -1 \) and that \( j^3 = -j \) and that \( j^4 = 1 \). Going beyond this, we see how \( j^5 = j \) and \( j^6 = -1 \) and \( j^7 = -j \) and \( j^8 = 1 \), etc.

Next, we will see what happens when we apply \( j \) as a multiplying coefficient in the sine series. This simply multiplies every term by \( j \):

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots
\]

\[
j \sin x = jx - \frac{x^3}{3!} + j \frac{x^5}{5!} - j \frac{x^7}{7!} + \cdots
\]

Finally, when we set these three equations beside each other, it is apparent how \( e^{jx} \) is the sum of \( \cos x \) and \( j \sin x \). This fact may be clearly shown by using different colors (blue and red) to represent the cosine and sine terms, respectively, and writing the exponential series as an alternating series of those blue and red terms taken from the cosine and sine series:

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots
\]

\[
j \sin x = jx - \frac{x^3}{3!} + j \frac{x^5}{5!} - j \frac{x^7}{7!} + \cdots
\]

\[
e^{jx} = 1 + jx - \frac{x^2}{2!} - j \frac{x^3}{3!} + \frac{x^4}{4!} + j \frac{x^5}{5!} - \frac{x^6}{6!} - j \frac{x^7}{7!} + \cdots
\]

And with that, we have Euler’s Relation:

\[
e^{jx} = \cos x + j \sin x
\]
Euler’s Relation is incredibly important in many different branches of engineering and science because it provides an equivalence between exponential functions which are relatively easy to manipulate using calculus, and trigonometric functions which are relatively difficult to manipulate using calculus. Therefore, any application requiring the application of calculus to sinusoidal functions can benefit from Euler’s Relation.
Chapter 5

Questions

This learning module, along with all others in the ModEL collection, is designed to be used in an inverted instructional environment where students independently read the tutorials and attempt to answer questions on their own prior to the instructor’s interaction with them. In place of lecture, the instructor engages with students in Socratic-style dialogue, probing and challenging their understanding of the subject matter through inquiry.

Answers are not provided for questions within this chapter, and this is by design. Solved problems may be found in the Tutorial and Derivation chapters, instead. The goal here is independence, and this requires students to be challenged in ways where others cannot think for them. Remember that you always have the tools of experimentation and computer simulation (e.g. SPICE) to explore concepts!

The following lists contain ideas for Socratic-style questions and challenges. Upon inspection, one will notice a strong theme of metacognition within these statements: they are designed to foster a regular habit of examining one’s own thoughts as a means toward clearer thinking. As such these sample questions are useful both for instructor-led discussions as well as for self-study.

---

1 Technical reading is an essential academic skill for any technical practitioner to possess for the simple reason that the most comprehensive, accurate, and useful information to be found for developing technical competence is in textual form. Technical careers in general are characterized by the need for continuous learning to remain current with standards and technology, and therefore any technical practitioner who cannot read well is handicapped in their professional development. An excellent resource for educators on improving students’ reading prowess through intentional effort and strategy is the book *Reading For Understanding – How Reading Apprenticeship Improves Disciplinary Learning in Secondary and College Classrooms* by Ruth Schoenbach, Cynthia Greenleaf, and Lynn Murphy.

2 Lecture is popular as a teaching method because it is easy to implement: any reasonably articulate subject matter expert can talk to students, even with little preparation. However, it is also quite problematic. A good lecture always makes complicated concepts seem easier than they are, which is bad for students because it instills a false sense of confidence in their own understanding; reading and re-articulation requires more cognitive effort and serves to verify comprehension. A culture of teaching-by-lecture fosters a debilitating dependence upon direct personal instruction, whereas the challenges of modern life demand independent and critical thought made possible only by gathering information and perspectives from afar. Information presented in a lecture is ephemeral, easily lost to failures of memory and dictation; text is forever, and may be referenced at any time.
CHAPTER 5. QUESTIONS

General challenges following tutorial reading

- **Summarize** as much of the text as you can in one paragraph of your own words. A helpful strategy is to explain ideas as you would for an intelligent child: as simple as you can without compromising too much accuracy.

- **Simplify** a particular section of the text, for example a paragraph or even a single sentence, so as to capture the same fundamental idea in fewer words.

- Where did the text **make the most sense** to you? What was it about the text’s presentation that made it clear?

- Identify where it might be easy for someone to **misunderstand the text**, and explain why you think it could be confusing.

- Identify any **new concept(s)** presented in the text, and explain in your own words.

- Identify any **familiar concept(s)** such as physical laws or principles applied or referenced in the text.

- Devise a **proof of concept** experiment demonstrating an important principle, physical law, or technical innovation represented in the text.

- Devise an experiment to **disprove** a plausible misconception.

- Did the text reveal any **misconceptions** you might have harbored? If so, describe the misconception(s) and the reason(s) why you now know them to be incorrect.

- Describe any useful **problem-solving strategies** applied in the text.

- **Devise a question** of your own to challenge a reader’s comprehension of the text.
General follow-up challenges for assigned problems

- Identify where any fundamental laws or principles apply to the solution of this problem, especially before applying any mathematical techniques.
- Devise a thought experiment to explore the characteristics of the problem scenario, applying known laws and principles to mentally model its behavior.
- Describe in detail your own strategy for solving this problem. How did you identify and organized the given information? Did you sketch any diagrams to help frame the problem?
- Is there more than one way to solve this problem? Which method seems best to you?
- Show the work you did in solving this problem, even if the solution is incomplete or incorrect.
- What would you say was the most challenging part of this problem, and why was it so?
- Was any important information missing from the problem which you had to research or recall?
- Was there any extraneous information presented within this problem? If so, what was it and why did it not matter?
- Examine someone else’s solution to identify where they applied fundamental laws or principles.
- Simplify the problem from its given form and show how to solve this simpler version of it. Examples include eliminating certain variables or conditions, altering values to simpler (usually whole) numbers, applying a limiting case (i.e. altering a variable to some extreme or ultimate value).
- For quantitative problems, identify the real-world meaning of all intermediate calculations: their units of measurement, where they fit into the scenario at hand. Annotate any diagrams or illustrations with these calculated values.
- For quantitative problems, try approaching it qualitatively instead, thinking in terms of “increase” and “decrease” rather than definite values.
- For qualitative problems, try approaching it quantitatively instead, proposing simple numerical values for the variables.
- Were there any assumptions you made while solving this problem? Would your solution change if one of those assumptions were altered?
- Identify where it would be easy for someone to go astray in attempting to solve this problem.
- Formulate your own problem based on what you learned solving this one.

General follow-up challenges for experiments or projects

- In what way(s) was this experiment or project easy to complete?
- Identify some of the challenges you faced in completing this experiment or project.
CHAPTER 5. QUESTIONS

- Show how thorough documentation assisted in the completion of this experiment or project.
- Which fundamental laws or principles are key to this system’s function?
- Identify any way(s) in which one might obtain false or otherwise misleading measurements from test equipment in this system.
- What will happen if (component X) fails (open/shorted/etc.)?
- What would have to occur to make this system unsafe?
5.1 Conceptual reasoning

These questions are designed to stimulate your analytic and synthetic thinking. In a Socratic discussion with your instructor, the goal is for these questions to prompt an extended dialogue where assumptions are revealed, conclusions are tested, and understanding is sharpened. Your instructor may also pose additional questions based on those assigned, in order to further probe and refine your conceptual understanding.

Questions that follow are presented to challenge and probe your understanding of various concepts presented in the tutorial. These questions are intended to serve as a guide for the Socratic dialogue between yourself and the instructor. Your instructor’s task is to ensure you have a sound grasp of these concepts, and the questions contained in this document are merely a means to this end. Your instructor may, at his or her discretion, alter or substitute questions for the benefit of tailoring the discussion to each student’s needs. The only absolute requirement is that each student is challenged and assessed at a level equal to or greater than that represented by the documented questions.

It is far more important that you convey your reasoning than it is to simply convey a correct answer. For this reason, you should refrain from researching other information sources to answer questions. What matters here is that you are doing the thinking. If the answer is incorrect, your instructor will work with you to correct it through proper reasoning. A correct answer without an adequate explanation of how you derived that answer is unacceptable, as it does not aid the learning or assessment process.

You will note a conspicuous lack of answers given for these conceptual questions. Unlike standard textbooks where answers to every other question are given somewhere toward the back of the book, here in these learning modules students must rely on other means to check their work. The best way by far is to debate the answers with fellow students and also with the instructor during the Socratic dialogue sessions intended to be used with these learning modules. Reasoning through challenging questions with other people is an excellent tool for developing strong reasoning skills.

Another means of checking your conceptual answers, where applicable, is to use circuit simulation software to explore the effects of changes made to circuits. For example, if one of these conceptual questions challenges you to predict the effects of altering some component parameter in a circuit, you may check the validity of your work by simulating that same parameter change within software and seeing if the results agree.

Analytical thinking involves the “disassembly” of an idea into its constituent parts, analogous to dissection. Synthetic thinking involves the “assembly” of a new idea comprised of multiple concepts, analogous to construction. Both activities are high-level cognitive skills, extremely important for effective problem-solving, necessitating frequent challenge and regular practice to fully develop.
5.1.1 Reading outline and reflections

“Reading maketh a full man; conference a ready man; and writing an exact man” – Francis Bacon

Francis Bacon’s advice is a blueprint for effective education: reading provides the learner with knowledge, writing focuses the learner’s thoughts, and critical dialogue equips the learner to confidently communicate and apply their learning. Independent acquisition and application of knowledge is a powerful skill, well worth the effort to cultivate. To this end, students should read these educational resources closely, write their own outline and reflections on the reading, and discuss in detail their findings with classmates and instructor(s). You should be able to do all of the following after reading any instructional text:

√ Briefly OUTLINE THE TEXT, as though you were writing a detailed Table of Contents. Feel free to rearrange the order if it makes more sense that way. Prepare to articulate these points in detail and to answer questions from your classmates and instructor. Outlining is a good self-test of thorough reading because you cannot outline what you have not read or do not comprehend.

√ Demonstrate ACTIVE READING STRATEGIES, including verbalizing your impressions as you read, simplifying long passages to convey the same ideas using fewer words, annotating text and illustrations with your own interpretations, working through mathematical examples shown in the text, cross-referencing passages with relevant illustrations and/or other passages, identifying problem-solving strategies applied by the author, etc. Technical reading is a special case of problem-solving, and so these strategies work precisely because they help solve any problem: paying attention to your own thoughts (metacognition), eliminating unnecessary complexities, identifying what makes sense, paying close attention to details, drawing connections between separated facts, and noting the successful strategies of others.

√ Identify IMPORTANT THEMES, especially GENERAL LAWS and PRINCIPLES, expounded in the text and express them in the simplest of terms as though you were teaching an intelligent child. This emphasizes connections between related topics and develops your ability to communicate complex ideas to anyone.

√ Form YOUR OWN QUESTIONS based on the reading, and then pose them to your instructor and classmates for their consideration. Anticipate both correct and incorrect answers, the incorrect answer(s) assuming one or more plausible misconceptions. This helps you view the subject from different perspectives to grasp it more fully.

√ Devise EXPERIMENTS to test claims presented in the reading, or to disprove misconceptions. Predict possible outcomes of these experiments, and evaluate their meanings: what result(s) would confirm, and what would constitute disproof? Running mental simulations and evaluating results is essential to scientific and diagnostic reasoning.

√ Specifically identify any points you found CONFUSING. The reason for doing this is to help diagnose misconceptions and overcome barriers to learning.
5.1. Conceptual Reasoning

5.1.2 Foundational concepts

Correct analysis and diagnosis of electric circuits begins with a proper understanding of some basic concepts. The following is a list of some important concepts referenced in this module’s tutorial. Define each of them in your own words, and be prepared to illustrate each of these concepts with a description of a practical example and/or a live demonstration.

- Energy
- Conservation of Energy
- Phasor
- Phase angle
5.1.3 Radians

A radian is defined as that angle describing a sector of a circle, whose arc length is equal to the radius of the circle:

Based on this definition, how many radians comprise a full circle?

How many degrees are in one radian?

Radians are considered a “natural” measurement unit for angles, while degrees are “arbitrary”. Explain why this is.

**Challenges**

- Calculate the angular velocity of a waveform having a frequency of 60 Hz (i.e. 60 cycles, or revolutions, per second).
5.2 Quantitative reasoning

These questions are designed to stimulate your computational thinking. In a Socratic discussion with your instructor, the goal is for these questions to reveal your mathematical approach(es) to problem-solving so that good technique and sound reasoning may be reinforced. Your instructor may also pose additional questions based on those assigned, in order to observe your problem-solving firsthand.

Mental arithmetic and estimations are strongly encouraged for all calculations, because without these abilities you will be unable to readily detect errors caused by calculator misuse (e.g. keystroke errors).

You will note a conspicuous lack of answers given for these quantitative questions. Unlike standard textbooks where answers to every other question are given somewhere toward the back of the book, here in these learning modules students must rely on other means to check their work. My advice is to use circuit simulation software such as SPICE to check the correctness of quantitative answers. Refer to those learning modules within this collection focusing on SPICE to see worked examples which you may use directly as practice problems for your own study, and/or as templates you may modify to run your own analyses and generate your own practice problems.

Completely worked example problems found in the Tutorial may also serve as “test cases” for gaining proficiency in the use of circuit simulation software, and then once that proficiency is gained you will never need to rely on an answer key!

5.2.1 Introduction to computer programming

A powerful tool for mathematical modeling is text-based computer programming. This is where you type coded commands in text form which the computer is able to interpret. Many different text-based languages exist for this purpose, but we will focus here on just two of them, C++ and Python.

---

4 In other words, set up the circuit simulation software to analyze the same circuit examples found in the Tutorial. If the simulated results match the answers shown in the Tutorial, it confirms the simulation has properly run. If the simulated results disagree with the Tutorial’s answers, something has been set up incorrectly in the simulation software. Using every Tutorial as practice in this way will quickly develop proficiency in the use of circuit simulation software.

5 This approach is perfectly in keeping with the instructional philosophy of these learning modules: teaching students to be self-sufficient thinkers. Answer keys can be useful, but it is even more useful to your long-term success to have a set of tools on hand for checking your own work, because once you have left school and are on your own, there will no longer be “answer keys” available for the problems you will have to solve.
CHAPTER 5. QUESTIONS

Programming in C++

One of the more popular text-based computer programming languages is called C++. This is a compiled language, which means you must create a plain-text file containing C++ code using a program called a text editor, then execute a software application called a compiler to translate your “source code” into instructions directly understandable to the computer. Here is an example of “source code” for a very simple C++ program intended to perform some basic arithmetic operations and print the results to the computer’s console:

```cpp
#include <iostream>
using namespace std;

int main ()
{
    float x, y;

    x = 200;
    y = -560.5;

    cout << "This simple program performs basic arithmetic on" << endl;
    cout << "the two numbers " << x << " and " << y << " and then" << endl;
    cout << "displays the results on the computer’s console." << endl;

    cout << "Sum = " << x + y << endl;
    cout << "Difference = " << x - y << endl;
    cout << "Product = " << x * y << endl;
    cout << "Quotient of " << x / y << endl;

    return 0;
}
```

Computer languages such as C++ are designed to make sense when read by human programmers. The general order of execution is left-to-right, top-to-bottom just the same as reading any text document written in English. Blank lines, indentation, and other “whitespace” is largely irrelevant in C++ code, and is included only to make the code more pleasing to view.

---

6 Although not included in this example, comments preceded by double-forward slash characters (`//`) may be added to source code as well to provide explanations of what the code is supposed to do, for the benefit of anyone reading it. The compiler application will ignore all comments.
Let's examine the C++ source code line by line to explain what it means:

- **#include <iostream>** and **using namespace std;** are set-up instructions to the compiler giving it some context in which to interpret your code. The code specific to your task is located between the brace symbols ({ and }).

- **int main()** labels the “Main” function for the computer: the instructions within this function (lying between the { and } symbols) it will be commanded to execute. Every complete C++ program contains a main function at minimum, and often additional functions as well. The int declares this function will return an integer number value when complete, which helps to explain the purpose of the return 0; statement at the end of the main function: providing a numerical value of zero at the program’s completion as promised by int. This returned value is rather incidental to our purpose here, but it is fairly standard practice in C++ programming.

- The float declaration reserves places in the computer’s memory for two floating-point variables, in this case the variables’ names being x and y.

- The next two lines assign numerical values to the two variables. Note how each line terminates with a semicolon character (;) and how this pattern holds true for most of the lines in this program.

- All the other instructions take the form of a cout command which prints characters to the “standard output” stream of the computer, which in this case will be text displayed on the console. The double-less-than symbols (<<) show data being sent toward the cout command. Note how verbatim text is enclosed in quotation marks, while variables such as x or mathematical expressions such as x - y are not enclosed in quotations because we want the computer to display the numerical values represented, not the literal text.

- The endl found at the end of every cout statement marks the end of a line of text. If not for these endl inclusions, the text displayed on the computer’s screen would resemble a run-on sentence rather than a paragraph. Note the cout << endl; line, which does nothing but create a blank line on the screen, for no reason other than esthetics.

After saving this source code text to a file with its own name (e.g. myprogram.cpp), you would then compile this source code into an executable file which the computer may then run. If you are using a console-based compiler such as GCC (very popular within variants of the Unix operating system7, such as Linux and Apple’s OS X), you would type the following command and press the Enter key:

```
g++ -o myprogram.exe myprogram.cpp
```

This command instructs the GCC compiler to take your source code (myprogram.cpp) and create with it an executable file named myprogram.exe. Simply typing ./myprogram.exe at the command-line will then execute your program:

```
./myprogram.exe
```

\(^7\) A very functional option for users of Microsoft Windows is called Cygwin, which provides a Unix-like console environment complete with all the customary utility applications such as GCC!
CHAPTER 5. QUESTIONS

If you are using a graphic-based C++ development system such as Microsoft Visual Studio\(^8\), you may simply create a new console application “project” using this software, then paste or type your code into the example template appearing in the editor window, and finally run your application to test its output.

As this program runs, it displays the following text to the console:

This simple program performs basic arithmetic on the two numbers 200 and -560.5 and then displays the results on the computer’s console.

- Sum = -360.5
- Difference = 760.5
- Product = -112100
- Quotient of -0.356824

As crude as this example program is, it serves the purpose of showing how easy it is to write and execute simple programs in a computer using the C++ language. As you encounter C++ example programs (shown as source code) in any of these modules, feel free to directly copy-and-paste the source code text into a text editor’s screen, then follow the rest of the instructions given here (i.e. save to a file, compile, and finally run your program). You will find that it is generally easier to learn computer programming by closely examining others’ example programs and modifying them than it is to write your own programs starting from a blank screen.

---

\(^8\)Using Microsoft Visual Studio community version 2017 at the time of this writing to test this example, here are the steps I needed to follow in order to successfully compile and run a simple program such as this: (1) Start up Visual Studio and select the option to create a New Project; (2) Select the Windows Console Application template, as this will perform necessary set-up steps to generate a console-based program which will save you time and effort as well as avoid simple errors of omission; (3) When the editing screen appears, type or paste the C++ code within the `main()` function provided in the template, deleting the “Hello World” `cout` line that came with the template; (4) Type or paste any preprocessor directives (e.g. `#include` statements, `namespace` statements) necessary for your code that did not come with the template; (5) Lastly, under the Debug drop-down menu choose either Start Debugging (F5 hot-key) or Start Without Debugging (Ctrl-F5 hotkeys) to compile (“Build”) and run your new program. Upon execution a console window will appear showing the output of your program.
5.2. QUANTITATIVE REASONING

Programming in Python

Another text-based computer programming language called Python allows you to type instructions at a terminal prompt and receive immediate results without having to compile that code. This is because Python is an interpreted language: a software application called an interpreter reads your source code, translates it into computer-understandable instructions, and then executes those instructions in one step.

The following shows what happens on my personal computer when I start up the Python interpreter on my personal computer, by typing python3 and pressing the Enter key:

```
Python 3.7.2 (default, Feb 19 2019, 18:15:18)
[GCC 4.1.2] on linux
Type "help", "copyright", "credits" or "license" for more information.
>>>
```

The >>> symbols represent the prompt within the Python interpreter “shell”, signifying readiness to accept Python commands entered by the user.

Shown here is an example of the same arithmetic operations performed on the same quantities, using a Python interpreter. All lines shown preceded by the >>> prompt are entries typed by the human programmer, and all lines shown without the >>> prompt are responses from the Python interpreter software:

```
>>> x = 200
>>> y = -560.5
>>> x + y
-360.5
>>> x - y
760.5
>>> x * y
-112100.0
>>> x / y
-0.35682426404995538
>>> quit()
```

9Using version 3 of Python, which is the latest at the time of this writing.
More advanced mathematical functions are accessible in Python by first entering the line `from math import *` which “imports” these functions from Python’s math library (included on any computer with Python installed). Some examples show some of these functions in use, demonstrating how the Python interpreter may be used as a scientific calculator:

```python
>>> from math import *
>>> sin(30.0)
-0.98803162409286183
>>> sin(radians(30.0))
0.49999999999999994
>>> pow(2.0, 5.0)
32.0
>>> log10(10000.0)
4.0
>>> e
2.7182818284590451
>>> pi
3.1415926535897931
>>> log(pow(e,6.0))
6.0
>>> asin(0.7071068)
0.78539819000368838
>>> degrees(asin(0.7071068))
45.000001524425265
>>> a = complex(3.0,4.0)
>>> b = complex(-2.8,10.5)
>>> a + b
(0.20000000000000018+14.5j)
>>> a - b
(5.8-6.5j)
>>> a * b
(-50.4+20.3j)
>>> a / b
(0.2845287492590397-0.36158861885002963j)
>>> quit()
```

Note how trigonometric functions assume angles expressed in radians rather than degrees, and how Python provides convenient functions for translating between the two. Logarithms assume a base of $e$ unless otherwise stated (e.g. the $\log_{10}$ function for common logarithms).

The interpreted (versus compiled) nature of Python, as well as its relatively simple syntax, makes it a good choice as a person’s first programming language. For complex applications, interpreted languages such as Python execute slower than compiled languages such as C++, but for the very simple examples used in these learning modules speed is not a concern.
Another Python math library is `cmath`, giving Python the ability to perform arithmetic on complex numbers. This is very useful for AC circuit analysis using phasors\(^{10}\) as shown in the following example. Here we see Python’s interpreter used as a scientific calculator to show series and parallel impedances of a resistor, capacitor, and inductor in a 60 Hz AC circuit:

```python
>>> from math import *
>>> from cmath import *
>>> r = complex(400,0)
>>> f = 60.0
>>> xc = 1/(2 * pi * f * 4.7e-6)
>>> zc = complex(0,-xc)
>>> xl = 2 * pi * f * 1.0
>>> zl = complex(0,xl)
>>> r + zc + zl
(400-187.38811239154882j)
>>> 1/(1/r + 1/zc + 1/zl)
(355.837695813625+125.35793777619385j)
>>> polar(r + zc + zl)
(441.717448903332, -0.4381072059213295)
>>> abs(r + zc + zl)
441.717448903332
>>> phase(r + zc + zl)
-0.4381072059213295
>>> degrees(phase(r + zc + zl))
-25.10169387356105
```

Note how Python defaults to rectangular form for complex quantities. Here we defined a 400 Ohm resistance as a complex value in rectangular form \((400 + j0 \ \Omega)\), then computed capacitive and inductive reactances at 60 Hz and defined each of those as complex (phasor) values \((0 - jX_c \ \Omega\) and \(0 + jX_l \ \Omega\), respectively). After that we computed total impedance in series, then total impedance in parallel. Polar-form representation was then shown for the series impedance \((441.717 \ \Omega \angle -25.102^\circ)\). Note the use of different functions to show the polar-form series impedance value: `polar()` takes the complex quantity and returns its polar magnitude and phase angle in radians; `abs()` returns just the polar magnitude; `phase()` returns just the polar angle, once again in radians. To find the polar phase angle in degrees, we nest the `degrees()` and `phase()` functions together.

The utility of Python’s interpreter environment as a scientific calculator should be clear from these examples. Not only does it offer a powerful array of mathematical functions, but also unlimited assignment of variables as well as a convenient text record\(^{11}\) of all calculations performed which may be easily copied and pasted into a text document for archival.

\(^{10}\)A “phasor” is a voltage, current, or impedance represented as a complex number, either in rectangular or polar form.

\(^{11}\)Like many command-line computing environments, Python’s interpreter supports “up-arrow” recall of previous entries. This allows quick recall of previously typed commands for editing and re-evaluation.
It is also possible to save a set of Python commands to a text file using a text editor application, and then instruct the Python interpreter to execute it at once rather than having to type it line-by-line in the interpreter's shell. For example, consider the following Python program, saved under the filename `myprogram.py`:

```python
x = 200
y = -560.5
print("Sum")
print(x + y)
print("Difference")
print(x - y)
print("Product")
print(x * y)
print("Quotient")
print(x / y)
```

As with C++, the interpreter will read this source code from left-to-right, top-to-bottom, just the same as you or I would read a document written in English. Interestingly, whitespace is significant in the Python language (unlike C++), but this simple example program makes no use of that.

To execute this Python program, I would need to type `python myprogram.py` and then press the Enter key at my computer console's prompt, at which point it would display the following result:

```
Sum
-360.5
Difference
760.5
Product
-112100.0
Quotient
-0.35682426405
```

As you can see, syntax within the Python programming language is simpler than C++, which is one reason why it is often a preferred language for beginning programmers.

If you are interested in learning more about computer programming in any language, you will find a wide variety of books and free tutorials available on those subjects. Otherwise, feel free to learn by the examples presented in these modules.
5.2.2 Simple phasor diagrams

Write both the rectangular and polar expressions for all phasors shown in this diagram:

\[ A = \]
\[ B = \]
\[ C = \]
\[ D = \]

Challenges

- ???.
- ???.
- ???.
- ???.

\[ A = \]
\[ B = \]
\[ C = \]
\[ D = \]
5.2.3 Practice: complex number calculations

These complex-number arithmetic problems are presented to you, complete with answers (shown in bold), for the purpose of practice, since nearly all AC circuit calculations will need to be performed using complex numbers. Use these practice calculations to check your ability either to perform these calculations “by hand” (using trigonometric functions) or your ability to use your calculator’s complex-number functionality.

Addition and subtraction:

\[
(5 + j6) + (2 - j1) = 7 + j5 \\
(10 - j8) + (4 - j3) = 14 - j11 \\
(-3 + j0) + (9 - j12) = 6 - j12
\]

\[
(3 + j5) - (0 - j9) = 3 + j14 \\
(25 - j84) - (4 - j3) = 21 - j81 \\
(-1500 + j40) + (299 - j128) = -1201 - j88
\]

\[
(25\angle15^\circ) + (10\angle74^\circ) = 31.35\angle30.87^\circ \\
(1000\angle43^\circ) + (1200\angle -20^\circ) = 1878.7\angle8.311^\circ \\
(522\angle71^\circ) - (85\angle30^\circ) = 461.23\angle77.94^\circ
\]

Multiplication and division:

\[
(25\angle15^\circ) \times (12\angle10^\circ) = 300\angle25^\circ \\
(1\angle25^\circ) \times (500\angle -30^\circ) = 500\angle -5^\circ \\
(522\angle71^\circ) \times (33\angle9^\circ) = 17226\angle80^\circ
\]

\[
\frac{10\angle -80^\circ}{1\angle0^\circ} = 10\angle -80^\circ \\
\frac{25\angle120^\circ}{3.5\angle-55^\circ} = 7.142\angle175^\circ \\
\frac{-66\angle67^\circ}{8\angle-42^\circ} = 8.25\angle -71^\circ
\]

\[
(3 + j5) \times (2 - j1) = 11 + j7 \\
(10 - j8) \times (4 - j3) = 16 - j62 \\
\frac{(3+j4)}{(12-j2)} = 0.1892 + j0.3649
\]
5.2. QUANTITATIVE REASONING

Reciprocation:

\[
\frac{1}{(15\angle 60^\circ)} = 0.0667\angle -60^\circ
\]

\[
\frac{1}{(750\angle -38^\circ)} = 0.00133\angle 38^\circ
\]

\[
\frac{1}{(10+3j)} = 0.0917 - j0.0275
\]

\[
\frac{1}{15\angle 45^\circ + 92\angle -25^\circ} = 14.06\angle 36.74^\circ
\]

\[
\frac{1}{1200\angle 73^\circ + 574\angle 21^\circ} = 425.7\angle 37.23^\circ
\]

\[
\frac{1}{238\angle -67^\circ + 108\angle -81^\circ} = 7.013k\angle -76.77^\circ
\]

\[
\frac{1}{110\angle -34^\circ + 80\angle 19^\circ + 70\angle 10} = 29.89\angle 2.513^\circ
\]

\[
\frac{1}{89\angle -5^\circ + 154\angle 33^\circ + 9353\angle 45} = 5.531k\angle 37.86^\circ
\]

\[
\frac{1}{512\angle 34^\circ + 18\angle -25^\circ + 9432\angle -20 + 2.28\angle 44^\circ} = 256.4\angle 9.181^\circ
\]
5.2.4 Series AC voltages

Determine the total voltage read between test points A and B by each of the AC voltmeters in these examples, as well as the same voltage in full complex (either polar or rectangular) notation. Assume all sources to be precisely the same frequency:

- **AC voltmeter**
  - 10 V $\angle$ 20°
  - 15 V $\angle$ 0°

- **AC voltmeter**
  - 120 V $\angle$ 0°
  - 50 V $\angle$ -30°

- **AC voltmeter**
  - $4 + 3j$ V
  - $8 - j2$ V

**Challenges**

- How would each of these answers differ, if at all, if test points A and B were reversed?
5.2.5 Parallel AC currents

Determine the total current read by each of the AC ammeters in these examples, as well as the same current in full complex (either polar or rectangular) notation. Assume all sources to be precisely the same frequency:

Challenges

- How would each of these answers differ, if at all, if the ammeter’s test leads were reversed?
5.3 Diagnostic reasoning

These questions are designed to stimulate your deductive and inductive thinking, where you must apply general principles to specific scenarios (deductive) and also derive conclusions about the failed circuit from specific details (inductive). In a Socratic discussion with your instructor, the goal is for these questions to reinforce your recall and use of general circuit principles and also challenge your ability to integrate multiple symptoms into a sensible explanation of what’s wrong in a circuit. Your instructor may also pose additional questions based on those assigned, in order to further challenge and sharpen your diagnostic abilities.

As always, your goal is to fully explain your analysis of each problem. Simply obtaining a correct answer is not good enough – you must also demonstrate sound reasoning in order to successfully complete the assignment. Your instructor’s responsibility is to probe and challenge your understanding of the relevant principles and analytical processes in order to ensure you have a strong foundation upon which to build further understanding.

You will note a conspicuous lack of answers given for these diagnostic questions. Unlike standard textbooks where answers to every other question are given somewhere toward the back of the book, here in these learning modules students must rely on other means to check their work. The best way by far is to debate the answers with fellow students and also with the instructor during the Socratic dialogue sessions intended to be used with these learning modules. Reasoning through challenging questions with other people is an excellent tool for developing strong reasoning skills.

Another means of checking your diagnostic answers, where applicable, is to use circuit simulation software to explore the effects of faults placed in circuits. For example, if one of these diagnostic questions requires that you predict the effect of an open or a short in a circuit, you may check the validity of your work by simulating that same fault (substituting a very high resistance in place of that component for an open, and substituting a very low resistance for a short) within software and seeing if the results agree.
5.3.1 Incorrect voltage calculation

A student learning about AC circuits for the first time constructs the following resistor-capacitor circuit in the lab:

![Resistor-Capacitor Circuit Diagram]

Measuring 3.75 Volts AC across the resistor and 5.05 Volts across the capacitor, the student concludes the source’s voltage must be 8.8 Volts AC. Identify the student’s error, and explain how they should have calculated total (source) voltage.

**Challenges**

- What fundamental principle of electric circuits did the student attempt to apply in adding $V_R$ and $V_C$? Is this principle still valid for AC circuits, or does it apply only to DC circuits?
Chapter 6

Projects and Experiments

The following project and experiment descriptions outline things you can build to help you understand circuits. With any real-world project or experiment there exists the potential for physical harm. *Electricity can be very dangerous in certain circumstances, and you should follow proper safety precautions at all times!*

### 6.1 Recommended practices

This section outlines some recommended practices for all circuits you design and construct.
CHAPTER 6. PROJECTS AND EXPERIMENTS

6.1.1 Safety first!

Electricity, when passed through the human body, causes uncomfortable sensations and in large enough measures\(^1\) will cause muscles to involuntarily contract. The overriding of your nervous system by the passage of electrical current through your body is particularly dangerous in regard to your heart, which is a vital muscle. Very large amounts of current can produce serious internal burns in addition to all the other effects.

Cardio-pulmonary resuscitation (CPR) is the standard first-aid for any victim of electrical shock. This is a very good skill to acquire if you intend to work with others on dangerous electrical circuits. You should never perform tests or work on such circuits unless someone else is present who is proficient in CPR.

As a general rule, any voltage in excess of 30 Volts poses a definitive electric shock hazard, because beyond this level human skin does not have enough resistance to safely limit current through the body. “Live” work of any kind with circuits over 30 volts should be avoided, and if unavoidable should only be done using electrically insulated tools and other protective equipment (e.g. insulating shoes and gloves). If you are unsure of the hazards, or feel unsafe at any time, stop all work and distance yourself from the circuit!

A policy I strongly recommend for students learning about electricity is to never come into electrical contact\(^2\) with an energized conductor, no matter what the circuit’s voltage\(^3\) level! Enforcing this policy may seem ridiculous when the circuit in question is powered by a single battery smaller than the palm of your hand, but it is precisely this instilled habit which will save a person from bodily harm when working with more dangerous circuits. Experience has taught me that students who learn early on to be careless with safe circuits have a tendency to be careless later with dangerous circuits!

In addition to the electrical hazards of shock and burns, the construction of projects and running of experiments often poses other hazards such as working with hand and power tools, potential

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\(^1\)Professor Charles Dalziel published a research paper in 1961 called “The Deleterious Effects of Electric Shock” detailing the results of electric shock experiments with both human and animal subjects. The threshold of perception for human subjects holding a conductor in their hand was in the range of 1 milliampere of current (less than this for alternating current, and generally less for female subjects than for male). Loss of muscular control was exhibited by half of Dalziel’s subjects at less than 10 milliamperes alternating current. Extreme pain, difficulty breathing, and loss of all muscular control occurred for over 99% of his subjects at direct currents less than 100 milliamperes and alternating currents less than 30 milliamperes. In summary, it doesn’t require much electric current to induce painful and even life-threatening effects in the human body! Your first and best protection against electric shock is maintaining an insulating barrier between your body and the circuit in question, such that current from that circuit will be unable to flow through your body.

\(^2\)By “electrical contact” I mean either directly touching an energized conductor with any part of your body, or indirectly touching it through a conductive tool. The only physical contact you should ever make with an energized conductor is via an electrically insulated tool, for example a screwdriver with an electrically insulated handle, or an insulated test probe for some instrument.

\(^3\)Another reason for consistently enforcing this policy, even on low-voltage circuits, is due to the dangers that even some low-voltage circuits harbor. A single 12 Volt automobile battery, for example, can cause a surprising amount of damage if short-circuited simply due to the high current levels (i.e. very low internal resistance) it is capable of, even though the voltage level is too low to cause a shock through the skin. Mechanics wearing metal rings, for example, are at risk from severe burns if their rings happen to short-circuit such a battery! Furthermore, even when working on circuits that are simply too low-power (low voltage and low current) to cause any bodily harm, touching them while energized can pose a threat to the circuit components themselves. In summary, it generally wise (and always a good habit to build) to “power down” any circuit before making contact between it and your body.
contact with high temperatures, potential chemical exposure, etc. You should never proceed with a project or experiment if you are unaware of proper tool use or lack basic protective measures (e.g. personal protective equipment such as safety glasses) against such hazards.

Some other safety-related practices should be followed as well:

- All power conductors extending outward from the project must be \textit{firmly} strain-relieved (e.g. “cord grips” used on line power cords), so that an accidental tug or drop will not compromise circuit integrity.

- All electrical connections must be sound and appropriately made (e.g. soldered wire joints rather than twisted-and-taped; terminal blocks rather than solderless breadboards for high-current or high-voltage circuits). Use “touch-safe” terminal connections with recessed metal parts to minimize risk of accidental contact.

- Always provide overcurrent protection in any circuit you build. \textit{Always}. This may be in the form of a fuse, a circuit breaker, and/or an electronically current-limited power supply.

- Always ensure circuit conductors are rated for more current than the overcurrent protection limit. \textit{Always}. A fuse does no good if the wire or printed circuit board trace will “blow” before it does!

- Always bond metal enclosures to Earth ground for any line-powered circuit. \textit{Always}. Ensuring an equipotential state between the enclosure and Earth by making the enclosure electrically common with Earth ground ensures no electric shock can occur simply by one’s body bridging between the Earth and the enclosure.

- Avoid building a high-energy circuit when a low-energy circuit will suffice. For example, I always recommend beginning students power their first DC resistor circuits using small batteries rather than with line-powered DC power supplies. The intrinsic energy limitations of a dry-cell battery make accidents highly unlikely.

- Use line power receptacles that are GFCI (Ground Fault Current Interrupting) to help avoid electric shock from making accidental contact with a “hot” line conductor.

- Always wear eye protection when working with tools or live systems having the potential to eject material into the air. Examples of such activities include soldering, drilling, grinding, cutting, wire stripping, working on or near energized circuits, etc.

- Always use a step-stool or stepladder to reach high places. Never stand on something not designed to support a human load.

- When in doubt, \textit{ask an expert}. If anything even seems remotely unsafe to you, do not proceed without consulting a trusted person fully knowledgeable in electrical safety.
6.1.2 Other helpful tips

Experience has shown the following practices to be very helpful, especially when students make their own component selections, to ensure the circuits will be well-behaved:

- **Avoid resistor values less than 1 kΩ or greater than 100 kΩ**, unless such values are definitely necessary. Resistances below 1 kΩ may draw excessive current if directly connected to a voltage source of significant magnitude, and may also complicate the task of accurately measuring current since any ammeter’s non-zero resistance inserted in series with a low-value circuit resistor will significantly alter the total resistance and thereby skew the measurement. Resistances above 100 kΩ may complicate the task of measuring voltage since any voltmeter’s finite resistance connected in parallel with a high-value circuit resistor will significantly alter the total resistance and thereby skew the measurement. Similarly, AC circuit impedance values should be between 1 kΩ and 100 kΩ, and for all the same reasons.

- **Ensure all electrical connections are low-resistance and physically rugged**. For this reason, one should avoid *compression splices* (e.g. “butt” connectors), solderless breadboards, and wires that are simply twisted together.

- **Build your circuit with testing in mind**. For example, provide convenient connection points for test equipment (e.g. multimeters, oscilloscopes, signal generators, logic probes).

- **Design permanent projects with maintenance in mind**. The more convenient you make maintenance tasks, the more likely they will get done.

- **Always document and save your work**. Circuits lacking schematic diagrams are more difficult to troubleshoot than documented circuits. Similarly, circuit construction is simpler when a schematic diagram precedes construction. Experimental results are easier to interpret when comprehensively recorded. Consider modern videorecording technology for this purpose where appropriate.

- **Record your steps** when troubleshooting. **Talk to yourself** when solving problems. These simple steps clarify thought and simplify identification of errors.

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4 An example of a necessary resistor value much less than 1 kΩ is a *shunt resistor* used to produce a small voltage drop for the purpose of sensing current in a circuit. Such shunt resistors must be low-value in order not to impose an undue load on the rest of the circuit. An example of a necessary resistor value much greater than 100 kΩ is an electrostatic *drain resistor* used to dissipate stored electric charges from body capacitance for the sake of preventing damage to sensitive semiconductor components, while also preventing a path for current that could be dangerous to the person (i.e. shock).

5 Admittedly, solderless breadboards are very useful for constructing complex electronic circuits with many components, especially DIP-style integrated circuits (ICs), but they tend to give trouble with connection integrity after frequent use. An alternative for projects using low counts of ICs is to solder IC sockets into prototype printed circuit boards (PCBs) and run wires from the soldered pins of the IC sockets to terminal blocks where reliable temporary connections may be made.
6.1. RECOMMENDED PRACTICES

6.1.3 Terminal blocks for circuit construction

Terminal blocks are the standard means for making electric circuit connections in industrial systems. They are also quite useful as a learning tool, and so I highly recommend their use in lieu of solderless breadboards. Terminal blocks provide highly reliable connections capable of withstanding significant voltage and current magnitudes, and they force the builder to think very carefully about component layout which is an important mental practice. Terminal blocks that mount on standard 35 mm DIN rail are made in a wide range of types and sizes, some with built-in disconnecting switches, some with built-in components such as rectifying diodes and fuseholders, all of which facilitate practical circuit construction.

I recommend every student of electricity build their own terminal block array for use in constructing experimental circuits, consisting of several terminal blocks where each block has at least 4 connection points all electrically common to each other and at least one terminal block that is a fuse holder for overcurrent protection. A pair of anchoring blocks hold all terminal blocks securely on the DIN rail, preventing them from sliding off the rail. Each of the terminals should bear a number, starting from 0. An example is shown in the following photograph and illustration:

![Diagram of terminal blocks and DIN rail](image)

Screwless terminal blocks (using internal spring clips to clamp wire and component lead ends) are preferred over screw-based terminal blocks, as they reduce assembly and disassembly time, and also minimize repetitive wrist stress from twisting screwdrivers. Some screwless terminal blocks require the use of a special tool to release the spring clip, while others provide buttons for this task which may be pressed using the tip of any suitable tool.

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6 Solderless breadboards are preferable for complicated electronic circuits with multiple integrated “chip” components, but for simpler circuits I find terminal blocks much more practical. An alternative to solderless breadboards for “chip” circuits is to solder chip sockets onto a PCB and then use wires to connect the socket pins to terminal blocks. This also accommodates surface-mount components, which solderless breadboards do not.

7 DIN rail is a metal rail designed to serve as a mounting point for a wide range of electrical and electronic devices such as terminal blocks, fuses, circuit breakers, relay sockets, power supplies, data acquisition hardware, etc.

8 Sometimes referred to as equipotential, same-potential, or potential distribution terminal blocks.

9 The small orange-colored squares seen in the above photograph are buttons for this purpose, and may be actuated by pressing with any tool of suitable size.
The following example shows how such a terminal block array might be used to construct a series-parallel resistor circuit consisting of four resistors and a battery:

Numbering on the terminal blocks provides a very natural translation to SPICE\textsuperscript{10} netlists, where component connections are identified by terminal number:

\begin{verbatim}
* Series-parallel resistor circuit
v1 1 0 dc 6
r1 2 5 7100
r2 5 8 2200
r3 2 8 3300
r4 8 11 4700
rjmp1 1 2 0.01
rjmp2 0 11 0.01
.op
.end
\end{verbatim}

Note the use of “jumper” resistances \texttt{rjmp1} and \texttt{rjmp2} to describe the wire connections between terminals 1 and 2 and between terminals 0 and 11, respectively. Being resistances, SPICE requires a resistance value for each, and here we see they have both been set to an arbitrarily low value of 0.01 Ohm realistic for short pieces of wire.

Listing all components and wires along with their numbered terminals happens to be a useful documentation method for any circuit built on terminal blocks, independent of SPICE. Such a “wiring sequence” may be thought of as a \textit{non-graphical description} of an electric circuit, and is exceptionally easy to follow.

\textsuperscript{10}SPICE is computer software designed to analyze electrical and electronic circuits. Circuits are described for the computer in the form of netlists which are text files listing each component type, connection node numbers, and component values.
6.1. RECOMMENDED PRACTICES

An example of a more elaborate terminal block array is shown in the following photograph, with terminal blocks and “ice-cube” style electromechanical relays mounted to DIN rail, which is turn mounted to a perforated subpanel. This “terminal block board” hosts an array of thirty-five undedicated terminal block sections, four SPDT toggle switches, four DPDT “ice-cube” relays, a step-down control power transformer, bridge rectifier and filtering capacitor, and several fuses for overcurrent protection:

Four plastic-bottomed “feet” support the subpanel above the benchtop surface, and an unused section of DIN rail stands ready to accept other components. Safety features include electrical bonding of the AC line power cord’s ground to the metal subpanel (and all metal DIN rails), mechanical strain relief for the power cord to isolate any cord tension from wire connections, clear plastic finger guards covering the transformer’s screw terminals, as well as fused overcurrent protection for the 120 Volt AC line power and the transformer’s 12 Volt AC output. The perforated holes happen to be on \( \frac{3}{4} \) inch centers, their presence making it very easy to attach other sections of DIN rail, or specialized electrical components, directly to the grounded metal subpanel. Such a “terminal block board” is an inexpensive yet highly flexible means to construct physically robust circuits using industrial wiring practices.

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11 An electrical subpanel is a thin metal plate intended for mounting inside an electrical enclosure. Components are attached to the subpanel, and the subpanel in turn bolts inside the enclosure. Subpanels allow circuit construction outside the confines of the enclosure, which speeds assembly. In this particular usage there is no enclosure, as the subpanel is intended to be used as an open platform for the convenient construction of circuits on a benchtop by students. In essence, this is a modern version of the traditional breadboard which was literally a wooden board such as might be used for cutting loaves of bread, but which early electrical and electronic hobbyists used as platforms for the construction of circuits.

12 At the time of this writing (2019) the cost to build this board is approximately $250 US dollars.
CHAPTER 6. PROJECTS AND EXPERIMENTS

6.1.4 Conducting experiments

An experiment is an exploratory act, a test performed for the purpose of assessing some proposition or principle. Experiments are the foundation of the scientific method, a process by which careful observation helps guard against errors of speculation. All good experiments begin with an hypothesis, defined by the American Heritage Dictionary of the English Language as:

An assertion subject to verification or proof, as (a) A proposition stated as a basis for argument or reasoning. (b) A premise from which a conclusion is drawn. (c) A conjecture that accounts, within a theory or ideational framework, for a set of facts and that can be used as a basis for further investigation.

Stated plainly, an hypothesis is an educated guess about cause and effect. The correctness of this initial guess matters little, because any well-designed experiment will reveal the truth of the matter. In fact, incorrect hypotheses are often the most valuable because the experiments they engender lead us to surprising discoveries. One of the beautiful aspects of science is that it is more focused on the process of learning than about the status of being correct. In order for an hypothesis to be valid, it must be testable, which means it must be a claim possible to refute given the right data. Hypotheses impossible to critique are useless.

Once an hypothesis has been formulated, an experiment must be designed to test that hypothesis. A well-designed experiment requires careful regulation of all relevant variables, both for personal safety and for prompting the hypothesized results. If the effects of one particular variable are to be tested, the experiment must be run multiple times with different values of (only) that particular variable. The experiment set up with the “baseline” variable set is called the control, while the experiment set up with different value(s) is called the test or experimental.

For some hypotheses a viable alternative to a physical experiment is a computer-simulated experiment or even a thought experiment. Simulations performed on a computer test the hypothesis against the physical laws encoded within the computer simulation software, and are particularly useful for students learning new principles for which simulation software is readily available.

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13Science is more about clarifying our view of the universe through a systematic process of error detection than it is about proving oneself to be right. Some scientists may happen to have large egos – and this may have more to do with the ways in which large-scale scientific research is funded than anything else – but scientific method itself is devoid of ego, and if embraced as a practical philosophy is quite an effective stimulant for humility. Within the education system, scientific method is particularly valuable for helping students break free of the crippling fear of being wrong. So much emphasis is placed in formal education on assessing correct retention of facts that many students are fearful of saying or doing anything that might be perceived as a mistake, and of course making mistakes (i.e. having one’s hypotheses disapproved by experiment) is an indispensable tool for learning. Introducing science in the classroom – real science characterized by individuals forming actual hypotheses and testing those hypotheses by experiment – helps students become self-directed learners.

14This is the principle of falsifiability: that a scientific statement has value only insofar as it is liable to disproof given the requisite experimental evidence. Any claim that is unfalsifiable – that is, a claim which can never be disproven by any evidence whatsoever – could be completely wrong and we could never know it.

15A very pertinent example of this is learning how to analyze electric circuits using simulation software such as SPICE. A typical experimental cycle would proceed as follows: (1) Find or invent a circuit to analyze; (2) Apply your analytical knowledge to that circuit, predicting all voltages, currents, powers, etc. relevant to the concepts you are striving to master; (3) Run a simulation on that circuit, collecting “data” from the computer when complete; (4) Evaluate whether or not your hypotheses (i.e. predicted voltages, currents, etc.) agree with the computer-generated results; (5) If so, your analyses are (provisionally) correct – if not, examine your analyses and the computer simulation again to determine the source of error; (6) Repeat this process as many times as necessary until you achieve mastery.
6.1. RECOMMENDED PRACTICES

Thought experiments are useful for detecting inconsistencies within your own understanding of some subject, rather than testing your understanding against physical reality.

Here are some general guidelines for conducting experiments:

- The clearer and more specific the hypothesis, the better. Vague or unfalsifiable hypotheses are useless because they will fit any experimental results, and therefore the experiment cannot teach you anything about the hypothesis.

- Collect as much data (i.e. information, measurements, sensory experiences) generated by an experiment as is practical. This includes the time and date of the experiment, too!

- Never discard or modify data gathered from an experiment. If you have reason to believe the data is unreliable, write notes to that effect, but never throw away data just because you think it is untrustworthy. It is quite possible that even “bad” data holds useful information, and that someone else may be able to uncover its value even if you do not.

- Prioritize quantitative data over qualitative data wherever practical. Quantitative data is more specific than qualitative, less prone to subjective interpretation on the part of the experimenter, and amenable to an arsenal of analytical methods (e.g. statistics).

- Guard against your own bias(es) by making your experimental results available to others. This allows other people to scrutinize your experimental design and collected data, for the purpose of detecting and correcting errors you may have missed. Document your experiment such that others may independently replicate it.

- Always be looking for sources of error. No physical measurement is perfect, and so it is impossible to achieve exact values for any variable. Quantify the amount of uncertainty (i.e. the “tolerance” of errors) whenever possible, and be sure your hypothesis does not depend on precision better than this!

- Always remember that scientific confirmation is provisional – no number of “successful” experiments will prove an hypothesis true for all time, but a single experiment can disprove it. Put into simpler terms, truth is elusive but error is within reach.

- Remember that scientific method is about learning, first and foremost. An unfortunate consequence of scientific triumph in modern society is that science is often viewed by non-practitioners as an unerring source of truth, when in fact science is an ongoing process of challenging existing ideas to probe for errors and oversights. This is why it is perfectly acceptable to have a failed hypothesis, and why the only truly failed experiment is one where nothing was learned.
The following is an example of a well-planned and executed experiment, in this case a physical experiment demonstrating Ohm’s Law.

Planning Time/Date = 09:30 on 12 February 2019

HYPOTHESIS: the current through any resistor should be exactly proportional to the voltage impressed across it.

PROCEDURE: connect a resistor rated 1 k Ohm and 1/4 Watt to a variable-voltage DC power supply. Use an ammeter in series to measure resistor current and a voltmeter in parallel to measure resistor voltage.

RISKS AND MITIGATION: excessive power dissipation may harm the resistor and/or pose a burn hazard, while excessive voltage poses an electric shock hazard. 30 Volts is a safe maximum voltage for laboratory practices, and according to Joule’s Law a 1000 Ohm resistor will dissipate 0.25 Watts at 15.81 Volts ($P = V^2 / R$), so I will remain below 15 Volts just to be safe.

Experiment Time/Date = 10:15 on 12 February 2019

DATA COLLECTED:

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<th>Current</th>
</tr>
</thead>
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<td>5.400 V</td>
<td>5.206 mA</td>
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<tr>
<td>10.00 V</td>
<td>9.643 mA</td>
</tr>
<tr>
<td>14.00 V</td>
<td>13.49 mA</td>
</tr>
</tbody>
</table>

Analysis Time/Date = 10:57 on 12 February 2019

ANALYSIS: current definitely increases with voltage, and although I expected exactly one milliAmpere per Volt the actual current was usually less than that. The voltage/current ratios ranged from a low of 1036.87 (at 8.1 Volts) to a high of 1037.81 (at 14 Volts), but this represents a variance of only −0.0365% to +0.0541% from the average, indicating a very consistent proportionality — results consistent with Ohm’s Law.

ERROR SOURCES: one major source of error is the resistor’s value itself. I did not measure it, but simply assumed color bands of brown-black-red meant exactly 1000 Ohms. Based on the data I think the true resistance is closer to 1037 Ohms. Another possible explanation is multimeter calibration error. However, neither explains the small positive and negative variances from the average. This might be due to electrical noise, a good test being to repeat the same experiment to see if the variances are the same or different. Noise should generate slightly different results every time.
The following is an example of a well-planned and executed virtual experiment, in this case demonstrating Ohm’s Law using a computer (SPICE) simulation.

Planning Time/Date = 12:32 on 14 February 2019

HYPOTHESIS: for any given resistor, the current through that resistor should be exactly proportional to the voltage impressed across it.

PROCEDURE: write a SPICE netlist with a single DC voltage source and single 1000 Ohm resistor, then use NGSPICE version 26 to perform a "sweep" analysis from 0 Volts to 25 Volts in 5 Volt increments.

* SPICE circuit
  v1 1 0 dc
  r1 1 0 1000
  .dc v1 0 25 5
  .print dc v(1) i(v1)
  .end

RISKS AND MITIGATION: none.

DATA COLLECTED:

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<th>v(1)</th>
<th>v1#branch</th>
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<td>2.5000000e+01</td>
<td>-2.5000000e-02</td>
</tr>
</tbody>
</table>

Analysis Time/Date = 13:06 on 14 February 2019

ANALYSIS: perfect agreement between data and hypothesis -- current is precisely 1/1000 of the applied voltage for all values. Anything other than perfect agreement would have probably meant my netlist was incorrect. The negative current values surprised me, but it seems this is just how SPICE interprets normal current through a DC voltage source.

ERROR SOURCES: none.
As gratuitous as it may seem to perform experiments on a physical law as well-established as Ohm’s Law, even the examples listed previously demonstrate opportunity for real learning. In the physical experiment example, the student should identify and explain why their data does not perfectly agree with the hypothesis, and this leads them naturally to consider sources of error. In the computer-simulated experiment, the student is struck by SPICE’s convention of denoting regular current through a DC voltage source as being negative in sign, and this is also useful knowledge for future simulations. Scientific experiments are most interesting when things do not go as planned!

Aside from verifying well-established physical laws, simple experiments are extremely useful as educational tools for a wide range of purposes, including:

- Component familiarization (e.g. Which terminals of this switch connect to the NO versus NC contacts?)
- System testing (e.g. How heavy of a load can my AC-DC power supply source before the semiconductor components reach their thermal limits?)
- Learning programming languages (e.g. Let’s try to set up an “up” counter function in this PLC!)

Above all, the priority here is to inculcate the habit of hypothesizing, running experiments, and analyzing the results. This experimental cycle not only serves as an excellent method for self-directed learning, but it also works exceptionally well for troubleshooting faults in complex systems, and for these reasons should be a part of every technician’s and every engineer’s education.

### 6.1.5 Constructing projects

Designing, constructing, and testing projects is a very effective means of practical education. Within a formal educational setting, projects are generally chosen (or at least vetted) by an instructor to ensure they may be reasonably completed within the allotted time of a course or program of study, and that they sufficiently challenge the student to learn certain important principles. In a self-directed environment, projects are just as useful as a learning tool but there is some risk of unwittingly choosing a project beyond one’s abilities, which can lead to frustration.

Here are some general guidelines for managing projects:

- Define your goal(s) before beginning a project: what do you wish to achieve in building it? What, exactly, should the completed project do?
- Analyze your project prior to construction. Document it in appropriate forms (e.g. schematic diagrams), predict its functionality, anticipate all associated risks. In other words, plan ahead.
- Set a reasonable budget for your project, and stay within it.
- Identify any deadlines, and set reasonable goals to meet those deadlines.
- Beware of scope creep: the tendency to modify the project’s goals before it is complete.
- Document your progress! An easy way to do this is to use photography or videography: take photos and/or videos of your project as it progresses. Document failures as well as successes, because both are equally valuable from the perspective of learning.
6.2 Experiment: (first experiment)
This is a description of the experiment!

<table>
<thead>
<tr>
<th>Challenges</th>
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6.3 Project: (first project)
This is a description of the project!

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Appendix A

Problem-Solving Strategies

The ability to solve complex problems is arguably one of the most valuable skills one can possess, and this skill is particularly important in any science-based discipline.

- **Study principles, not procedures.** Don’t be satisfied with merely knowing how to compute solutions – learn why those solutions work.

- **Identify** what it is you need to solve, identify all relevant data, identify all units of measurement, identify any general principles or formulae linking the given information to the solution, and then identify any “missing pieces” to a solution. **Annotate** all diagrams with this data.

- **Sketch a diagram** to help visualize the problem. When building a real system, always devise a plan for that system and analyze its function **before** constructing it.

- **Follow the units of measurement and meaning of every calculation.** If you are ever performing mathematical calculations as part of a problem-solving procedure, and you find yourself unable to apply each and every intermediate result to some aspect of the problem, it means you don’t understand what you are doing. Properly done, every mathematical result should have practical meaning for the problem, and not just be an abstract number. You should be able to identify the proper units of measurement for each and every calculated result, and show where that result fits into the problem.

- **Perform “thought experiments”** to explore the effects of different conditions for theoretical problems. When troubleshooting real systems, perform **diagnostic tests** rather than visually inspecting for faults, the best diagnostic test being the one giving you the most information about the nature and/or location of the fault with the fewest steps.

- **Simplify the problem** until the solution becomes obvious, and then use that obvious case as a model to follow in solving the more complex version of the problem.

- **Check for exceptions** to see if your solution is incorrect or incomplete. A good solution will work for all known conditions and criteria. A good example of this is the process of testing scientific hypotheses: the task of a scientist is not to find support for a new idea, but rather to **challenge** that new idea to see if it holds up under a battery of tests. The philosophical
principle of *reductio ad absurdum* (i.e. disproving a general idea by finding a specific case where it fails) is useful here.

- **Work “backward”** from a hypothetical solution to a new set of given conditions.
- **Add quantities** to problems that are qualitative in nature, because sometimes a little math helps illuminate the scenario.
- **Sketch graphs** illustrating how variables relate to each other. These may be quantitative (i.e. with realistic number values) or qualitative (i.e. simply showing increases and decreases).
- **Treat quantitative problems as qualitative** in order to discern the relative magnitudes and/or directions of change of the relevant variables. For example, try determining what happens if a certain variable were to increase or decrease before attempting to precisely calculate quantities: how will each of the dependent variables respond, by increasing, decreasing, or remaining the same as before?
- **Consider limiting cases.** This works especially well for qualitative problems where you need to determine which direction a variable will change. Take the given condition and magnify that condition to an extreme degree as a way of simplifying the direction of the system’s response.
- **Check your work.** This means regularly testing your conclusions to see if they make sense. This does *not* mean repeating the same steps originally used to obtain the conclusion(s), but rather to use some other means to check validity. Simply repeating procedures often leads to *repeating the same errors* if any were made, which is why alternative paths are better.
Appendix B

Instructional philosophy

“The unexamined circuit is not worth energizing” – Socrates (if he had taught electricity)

These learning modules, although useful for self-study, were designed to be used in a formal learning environment where a subject-matter expert challenges students to digest the content and exercise their critical thinking abilities in the answering of questions and in the construction and testing of working circuits.

The following principles inform the instructional and assessment philosophies embodied in these learning modules:

• The first goal of education is to enhance clear and independent thought, in order that every student reach their fullest potential in a highly complex and inter-dependent world. Robust reasoning is always more important than particulars of any subject matter, because its application is universal.

• Literacy is fundamental to independent learning and thought because text continues to be the most efficient way to communicate complex ideas over space and time. Those who cannot read with ease are limited in their ability to acquire knowledge and perspective.

• Articulate communication is fundamental to work that is complex and interdisciplinary.

• Faulty assumptions and poor reasoning are best corrected through challenge, not presentation. The rhetorical technique of *reductio ad absurdum* (disproving an assertion by exposing an absurdity) works well to discipline student’s minds, not only to correct the problem at hand but also to learn how to detect and correct future errors.

• Important principles should be repeatedly explored and widely applied throughout a course of study, not only to reinforce their importance and help ensure their mastery, but also to showcase the interconnectedness and utility of knowledge.
These learning modules were expressly designed to be used in an “inverted” teaching environment¹ where students first read the introductory and tutorial chapters on their own, then individually attempt to answer the questions and construct working circuits according to the experiment and project guidelines. The instructor never lectures, but instead meets regularly with each individual student to review their progress, answer questions, identify misconceptions, and challenge the student to new depths of understanding through further questioning. Regular meetings between instructor and student should resemble a Socratic² dialogue, where questions serve as scalpels to dissect topics and expose assumptions. The student passes each module only after consistently demonstrating their ability to logically analyze and correctly apply all major concepts in each question or project/experiment. The instructor must be vigilant in probing each student’s understanding to ensure they are truly reasoning and not just memorizing. This is why “Challenge” points appear throughout, as prompts for students to think deeper about topics and as starting points for instructor queries. Sometimes these challenge points require additional knowledge that hasn’t been covered in the series to answer in full. This is okay, as the major purpose of the Challenges is to stimulate analysis and synthesis on the part of each student.

The instructor must possess enough mastery of the subject matter and awareness of students' reasoning to generate their own follow-up questions to practically any student response. Even completely correct answers given by the student should be challenged by the instructor for the purpose of having students practice articulating their thoughts and defending their reasoning. Conceptual errors committed by the student should be exposed and corrected not by direct instruction, but rather by reducing the errors to an absurdity³ through well-chosen questions and thought experiments posed by the instructor. Becoming proficient at this style of instruction requires time and dedication, but the positive effects on critical thinking for both student and instructor are spectacular.

An inspection of these learning modules reveals certain unique characteristics. One of these is a bias toward thorough explanations in the tutorial chapters. Without a live instructor to explain concepts and applications to students, the text itself must fulfill this role. This philosophy results in lengthier explanations than what you might typically find in a textbook, each step of the reasoning process fully explained, including footnotes addressing common questions and concerns students raise while learning these concepts. Each tutorial seeks to not only explain each major concept in sufficient detail, but also to explain the logic of each concept and how each may be developed

¹In a traditional teaching environment, students first encounter new information via lecture from an expert, and then independently apply that information via homework. In an “inverted” course of study, students first encounter new information via homework, and then independently apply that information under the scrutiny of an expert. The expert’s role in lecture is to simply explain, but the expert’s role in an inverted session is to challenge, critique, and if necessary explain where gaps in understanding still exist.

²Socrates is a figure in ancient Greek philosophy famous for his unflinching style of questioning. Although he authored no texts, he appears as a character in Plato’s many writings. The essence of Socratic philosophy is to leave no question unexamined and no point of view unchallenged. While purists may argue a topic such as electric circuits is too narrow for a true Socratic-style dialogue, I would argue that the essential thought processes involved with scientific reasoning on any topic are not far removed from the Socratic ideal, and that students of electricity and electronics would do very well to challenge assumptions, pose thought experiments, identify fallacies, and otherwise employ the arsenal of critical thinking skills modeled by Socrates.

³This rhetorical technique is known by the Latin phrase reductio ad absurdum. The concept is to expose errors by counter-example, since only one solid counter-example is necessary to disprove a universal claim. As an example of this, consider the common misconception among beginning students of electricity that voltage cannot exist without current. One way to apply reductio ad absurdum to this statement is to ask how much current passes through a fully-charged battery connected to nothing (i.e. a clear example of voltage existing without current).
from “first principles”. Again, this reflects the goal of developing clear and independent thought in students’ minds, by showing how clear and logical thought was used to forge each concept. Students benefit from witnessing a model of clear thinking in action, and these tutorials strive to be just that.

Another characteristic of these learning modules is a lack of step-by-step instructions in the Project and Experiment chapters. Unlike many modern workbooks and laboratory guides where step-by-step instructions are prescribed for each experiment, these modules take the approach that students must learn to closely read the tutorials and apply their own reasoning to identify the appropriate experimental steps. Sometimes these steps are plainly declared in the text, just not as a set of enumerated points. At other times certain steps are implied, an example being assumed competence in test equipment use where the student should not need to be told again how to use their multimeter because that was thoroughly explained in previous lessons. In some circumstances no steps are given at all, leaving the entire procedure up to the student.

This lack of prescription is not a flaw, but rather a feature. Close reading and clear thinking are foundational principles of this learning series, and in keeping with this philosophy all activities are designed to require those behaviors. Some students may find the lack of prescription frustrating, because it demands more from them than what their previous educational experiences required. This frustration should be interpreted as an unfamiliarity with autonomous thinking, a problem which must be corrected if the student is ever to become a self-directed learner and effective problem-solver. Ultimately, the need for students to read closely and think clearly is more important both in the near-term and far-term than any specific facet of the subject matter at hand. If a student takes longer than expected to complete a module because they are forced to outline, digest, and reason on their own, so be it. The future gains enjoyed by developing this mental discipline will be well worth the additional effort and delay.

Another feature of these learning modules is that they do not treat topics in isolation. Rather, important concepts are introduced early in the series, and appear repeatedly as stepping-stones toward other concepts in subsequent modules. This helps to avoid the “compartmentalization” of knowledge, demonstrating the inter-connectedness of concepts and simultaneously reinforcing them. Each module is fairly complete in itself, reserving the beginning of its tutorial to a review of foundational concepts.

This methodology of assigning text-based modules to students for digestion and then using Socratic dialogue to assess progress and hone students’ thinking was developed over a period of several years by the author with his Electronics and Instrumentation students at the two-year college level. While decidedly unconventional and sometimes even unsettling for students accustomed to a more passive lecture environment, this instructional philosophy has proven its ability to convey conceptual mastery, foster careful analysis, and enhance employability so much better than lecture that the author refuses to ever teach by lecture again.

Problems which often go undiagnosed in a lecture environment are laid bare in this “inverted” format where students must articulate and logically defend their reasoning. This, too, may be unsettling for students accustomed to lecture sessions where the instructor cannot tell for sure who comprehends and who does not, and this vulnerability necessitates sensitivity on the part of the “inverted” session instructor in order that students never feel discouraged by having their errors exposed. Everyone makes mistakes from time to time, and learning is a lifelong process! Part of the instructor’s job is to build a culture of learning among the students where errors are not seen as shameful, but rather as opportunities for progress.
To this end, instructors managing courses based on these modules should adhere to the following principles:

• Student questions are always welcome and demand thorough, honest answers. The only type of question an instructor should refuse to answer is one the student should be able to easily answer on their own. Remember, the fundamental goal of education is for each student to learn to think clearly and independently. This requires hard work on the part of the student, which no instructor should ever circumvent. Anything done to bypass the student’s responsibility to do that hard work ultimately limits that student’s potential and thereby does real harm.

• It is not only permissible, but encouraged, to answer a student’s question by asking questions in return, these follow-up questions designed to guide the student to reach a correct answer through their own reasoning.

• All student answers demand to be challenged by the instructor and/or by other students. This includes both correct and incorrect answers – the goal is to practice the articulation and defense of one’s own reasoning.

• No reading assignment is deemed complete unless and until the student demonstrates their ability to accurately summarize the major points in their own terms. Recitation of the original text is unacceptable. This is why every module contains an “Outline and reflections” question as well as a “Foundational concepts” question in the Conceptual reasoning section, to prompt reflective reading.

• No assigned question is deemed answered unless and until the student demonstrates their ability to consistently and correctly apply the concepts to variations of that question. This is why module questions typically contain multiple “Challenges” suggesting different applications of the concept(s) as well as variations on the same theme(s). Instructors are encouraged to devise as many of their own “Challenges” as they are able, in order to have a multitude of ways ready to probe students’ understanding.

• No assigned experiment or project is deemed complete unless and until the student demonstrates the task in action. If this cannot be done “live” before the instructor, video-recordings showing the demonstration are acceptable. All relevant safety precautions must be followed, all test equipment must be used correctly, and the student must be able to properly explain all results. The student must also successfully answer all Challenges presented by the instructor for that experiment or project.
Students learning from these modules would do well to abide by the following principles:

- No text should be considered fully and adequately read unless and until you can express every idea in your own words, using your own examples.

- You should always articulate your thoughts as you read the text, noting points of agreement, confusion, and epiphanies. Feel free to print the text on paper and then write your notes in the margins. Alternatively, keep a journal for your own reflections as you read. This is truly a helpful tool when digesting complicated concepts.

- Never take the easy path of highlighting or underlining important text. Instead, summarize and/or comment on the text using your own words. This actively engages your mind, allowing you to more clearly perceive points of confusion or misunderstanding on your own.

- A very helpful strategy when learning new concepts is to place yourself in the role of a teacher, if only as a mental exercise. Either explain what you have recently learned to someone else, or at least imagine yourself explaining what you have learned to someone else. The simple act of having to articulate new knowledge and skill forces you to take on a different perspective, and will help reveal weaknesses in your understanding.

- Perform each and every mathematical calculation and thought experiment shown in the text on your own, referring back to the text to see that your results agree. This may seem trivial and unnecessary, but it is critically important to ensuring you actually understand what is presented, especially when the concepts at hand are complicated and easy to misunderstand. Apply this same strategy to become proficient in the use of circuit simulation software, checking to see if your simulated results agree with the results shown in the text.

- Above all, recognize that learning is hard work, and that a certain level of frustration is unavoidable. There are times when you will struggle to grasp some of these concepts, and that struggle is a natural thing. Take heart that it will yield with persistent and varied effort, and never give up!

Students interested in using these modules for self-study will also find them beneficial, although the onus of responsibility for thoroughly reading and answering questions will of course lie with that individual alone. If a qualified instructor is not available to challenge students, a workable alternative is for students to form study groups where they challenge one another.

To high standards of education,

Tony R. Kuphaldt

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4As the old saying goes, “Insanity is trying the same thing over and over again, expecting different results.” If you find yourself stumped by something in the text, you should attempt a different approach. Alter the thought experiment, change the mathematical parameters, do whatever you can to see the problem in a slightly different light, and then the solution will often present itself more readily.

5Avoid the temptation to simply share answers with study partners, as this is really counter-productive to learning. Always bear in mind that the answer to any question is far less important in the long run than the method(s) used to obtain that answer. The goal of education is to empower one’s life through the improvement of clear and independent thought, literacy, expression, and various practical skills.
Appendix C

Tools used

I am indebted to the developers of many open-source software applications in the creation of these learning modules. The following is a list of these applications with some commentary on each.

You will notice a theme common to many of these applications: a bias toward code. Although I am by no means an expert programmer in any computer language, I understand and appreciate the flexibility offered by code-based applications where the user (you) enters commands into a plain ASCII text file, which the software then reads and processes to create the final output. Code-based computer applications are by their very nature extensible, while WYSIWYG (What You See Is What You Get) applications are generally limited to whatever user interface the developer makes for you.

The GNU/Linux computer operating system

There is so much to be said about Linus Torvalds’ Linux and Richard Stallman’s GNU project. First, to credit just these two individuals is to fail to do justice to the mob of passionate volunteers who contributed to make this amazing software a reality. I first learned of Linux back in 1996, and have been using this operating system on my personal computers almost exclusively since then. It is free, it is completely configurable, and it permits the continued use of highly efficient Unix applications and scripting languages (e.g. shell scripts, Makefiles, sed, awk) developed over many decades. Linux not only provided me with a powerful computing platform, but its open design served to inspire my life’s work of creating open-source educational resources.

Bram Moolenaar’s Vim text editor

Writing code for any code-based computer application requires a text editor, which may be thought of as a word processor strictly limited to outputting plain-ASCII text files. Many good text editors exist, and one’s choice of text editor seems to be a deeply personal matter within the programming world. I prefer Vim because it operates very similarly to vi which is ubiquitous on Unix/Linux operating systems, and because it may be entirely operated via keyboard (i.e. no mouse required) which makes it fast to use.
Donald Knuth’s \TeX\ typesetting system

Developed in the late 1970’s and early 1980’s by computer scientist extraordinaire Donald Knuth to typeset his multi-volume magnum opus *The Art of Computer Programming*, this software allows the production of formatted text for screen-viewing or paper printing, all by writing plain-text code to describe how the formatted text is supposed to appear. \TeX\ is not just a markup language for documents, but it is also a Turing-complete programming language in and of itself, allowing useful algorithms to be created to control the production of documents. Simply put, \TeX\ *is a programmer’s approach to word processing*. Since \TeX\ is controlled by code written in a plain-text file, this means anyone may read that plain-text file to see exactly how the document was created. This openness afforded by the code-based nature of \TeX\ makes it relatively easy to learn how other people have created their own \TeX\ documents. By contrast, examining a beautiful document created in a conventional WYSIWYG word processor such as Microsoft *Word* suggests nothing to the reader about *how* that document was created, or what the user might do to create something similar. As Mr. Knuth himself once quipped, conventional word processing applications should be called WYSIAYG (What You See Is All You Get).

Leslie Lamport’s \LaTeX\ extensions to \TeX\n
Like all true programming languages, \TeX\ is inherently extensible. So, years after the release of \TeX\ to the public, Leslie Lamport decided to create a massive extension allowing easier compilation of book-length documents. The result was \LaTeX, which is the markup language used to create all ModEL module documents. You could say that \TeX\ is to \LaTeX\ as \C\ is to \C++. This means it is permissible to use any and all \TeX\ commands within \LaTeX\ source code, and it all still works. Some of the features offered by \LaTeX\ that would be challenging to implement in \TeX\ include automatic index and table-of-content creation.

Tim Edwards’ \Xcircuit\ drafting program

This wonderful program is what I use to create all the schematic diagrams and illustrations (but not photographic images or mathematical plots) throughout the ModEL project. It natively outputs PostScript format which is a true vector graphic format (this is why the images do not pixellate when you zoom in for a closer view), and it is so simple to use that I have never had to read the manual! Object libraries are easy to create for \Xcircuit, being plain-text files using PostScript programming conventions. Over the years I have collected a large set of object libraries useful for drawing electrical and electronic schematics, pictorial diagrams, and other technical illustrations.
Gimp graphic image manipulation program

Essentially an open-source clone of Adobe’s PhotoShop, I use Gimp to resize, crop, and convert file formats for all of the photographic images appearing in the ModEL modules. Although Gimp does offer its own scripting language (called Script-Fu), I have never had occasion to use it. Thus, my utilization of Gimp to merely crop, resize, and convert graphic images is akin to using a sword to slice a loaf of bread.

SPICE circuit simulation program

SPICE is to circuit analysis as TeX is to document creation: it is a form of markup language designed to describe a certain object to be processed in plain-ASCII text. When the plain-text “source file” is compiled by the software, it outputs the final result. More modern circuit analysis tools certainly exist, but I prefer SPICE for the following reasons: it is free, it is fast, it is reliable, and it is a fantastic tool for teaching students of electricity and electronics how to write simple code. I happen to use rather old versions of SPICE, version 2g6 being my “go to” application when I only require text-based output. NGSPICE (version 26), which is based on Berkeley SPICE version 3f5, is used when I require graphical output for such things as time-domain waveforms and Bode plots. In all SPICE example netlists I strive to use coding conventions compatible with all SPICE versions.

Andrew D. Hwang’s ePiX mathematical visualization programming library

This amazing project is a C++ library you may link to any C/C++ code for the purpose of generating PostScript graphic images of mathematical functions. As a completely free and open-source project, it does all the plotting I would otherwise use a Computer Algebra System (CAS) such as Mathematica or Maple to do. It should be said that ePiX is not a Computer Algebra System like Mathematica or Maple, but merely a mathematical visualization tool. In other words, it won’t determine integrals for you (you’ll have to implement that in your own C/C++ code!), but it can graph the results, and it does so beautifully. What I really admire about ePiX is that it is a C++ programming library, which means it builds on the existing power and toolset available with that programming language. Mr. Hwang could have probably developed his own stand-alone application for mathematical plotting, but by creating a C++ library to do the same thing he accomplished something much greater.
Appendix D

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Appendix E

Version history

This is a list showing all significant additions, corrections, and other edits made to this learning module. Each entry is referenced by calendar date in reverse chronological order (newest version first), which appears on the front cover of every learning module for easy reference. Any contributors to this open-source document are listed here as well.

23 May 2019 – added some clarifying comments to the Full Tutorial, and added more questions.

22 May 2019 – added questions.

21 May 2019 – added Derivation on Euler’s Relation, showing how $e^{jx} = \cos x + j \sin x$.

August 2018 – added content to the Introduction.

June 2018 – edited oscilloscope image to show triggering slope.

May 2018 – added a Simplified Tutorial chapter.

September 2017 – document first created.
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