April, 1992
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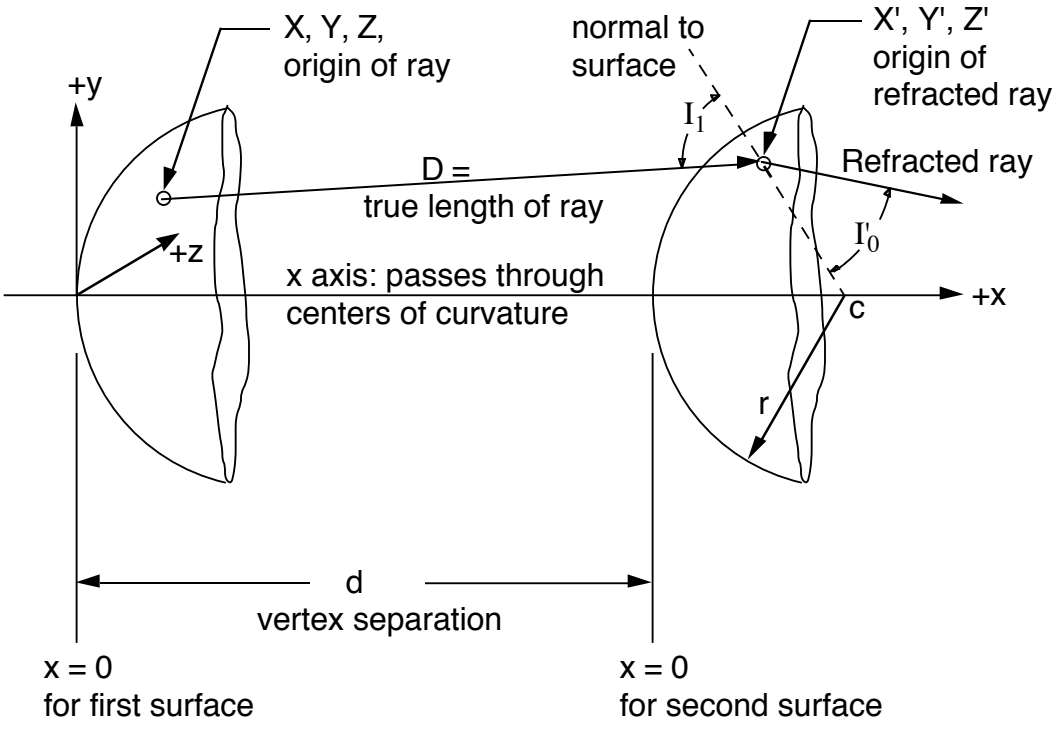
RAY TRACING NOTES

Ray tracing as given by Rudolf Kingslake in "Lens Design Fundamentals" Academic Press and "Lens Design" in Volume III of " Applied Optics and Engineering" Academic Press 1965; reference: M. Born and E. Wolf, "The Principles of Optics," 2nd ed., pp. 194-196. Macmillan (Pergamon), New York, 1964.

SECTION I - Finding X', Y', & Z', the point of intersection of a ray with a refracting surface.



The direction cosines for the ray of length D are K, L, and M.
The direction cosines for the refracted ray are K', L', and M'.
Neither I_1 nor I'_0 is necessarily in the plane of the paper.

FIGURE 1

Notes:

X dimensions are always referred to the vertices of the surfaces they're on.

To start this whole process, you must be given an initial X, Y, Z and K, L, M.

Direction cosines:

$$K = \frac{X' - X + d}{D} \qquad L = \frac{Y' - Y}{D} \qquad M = \frac{Z' - Z}{D}$$

From which:

$$X' = KD + (X - d) \qquad Y' = LD + Y \qquad Z' = MD + Z \qquad \text{(equa's. 1)}$$

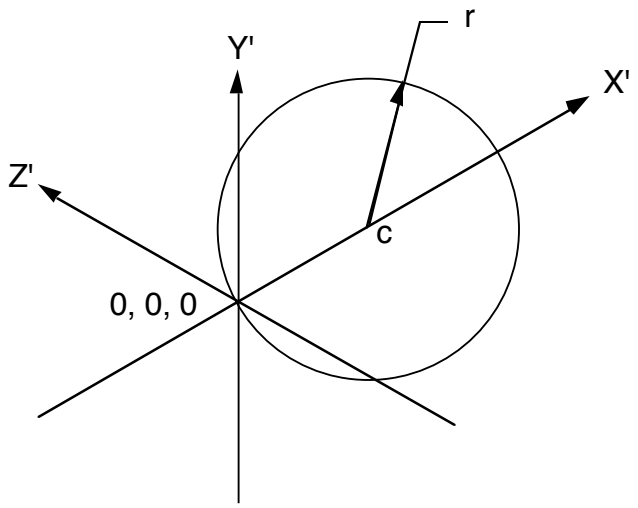


FIGURE 2

The equation for the sphere in Figure 2 is:

$$(X')^2 + (Y')^2 + (Z')^2 - 2\,r\,X' = 0 \qquad \text{(equa. 2)}$$

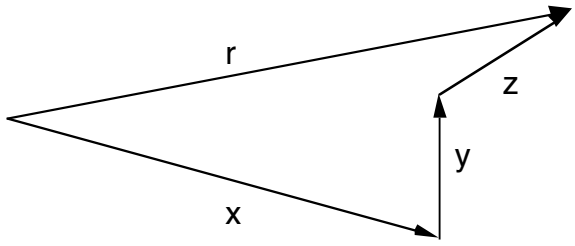
Note: If "c" were to the left of the origin, the equa. would be ....+2rX'. The easiest "fix" for this problem is to give "r" a negative sign if "c" is to the left of 0, 0, 0.

Substituting equations (1) into equation (2):

$$[\,K^2D^2 + 2KD(X - d) + (X - d)^2\,] + [\,L^2D^2 + 2LDY + Y^2\,] +$$
$$[\,M^2D^2 + 2MDZ + Z^2\,] - 2r\,[\,KD + (X - d)\,] = 0$$

Putting all equal powers of D together:
$(K^2 + L^2 + M^2)\,D^2 + \ldots\ldots = 0$

But, the sum of the squares of the direction cosines is one:



IF: r =1
THEN: x = K
y = L
z = M
AND:
$K^2 + L^2 + M^2 = 1$

FIGURE 3

Continuing with putting all equal powers of D together:

$D^2 + 2D [K(X - d) + LY + MZ - rK] + (X - d)^2 + Y^2 + Z^2 - 2r(X - d) = 0$

Or:

$D^2 - 2D [Kr - K(X - d) - LY - MZ_0] + (X - d)^2 + Y^2 + Z^2 - 2r (X - d) = 0$

Let:

$F = Kr - K(X - d) - LY - MZ$ (Equa. 3)

$G = (X - d)^2 + Y^2 + Z^2 - 2r(X - d)$ (Equa. 4)

NB: My F and G are different from Kingslake's; he divides my F and G by r to get his F and G. Then he uses rF and rG in the quadratic below where I use F and G. Why he does this is a mystery to me, but the answer may be in Born and Wolf's book.

Then:

$D^2 - 2FD + G = 0$

which is a quadratic

Solving the quadratic:

$D = [2F \pm (4F^2 - 4G)^{1/2}] / 2$

$D = F \pm (F^2 - G)^{1/2}$ (Equa. 5)

The "±" sign indicates the ray pierces the sphere at two points. In choosing the sense of the "±" sign, look at the configuration. Remember that D must always be positive. D will be the smaller of the two numbers if the situation is as shown in Figure 1. D = d on the optic axis.

For a plane surface,

$D = (d - X) / K$ because x' in equa. 1 becomes zero (Equa. 5p)

Conclusion of Section I:

In order to find X', Y', & Z'

1) Find D by
    a) Solving equa's. 3&4 and thence equa. 5
    b) Or: Solve equa. 5p
2) Solve equa's. 1 by plugging in value of D found above.

SECTION II - Finding K', L', M', the direction cosines of the refracted ray.

Part a - Finding cosine of the angle of incidence, I_1. This is the angle between the original ray and the normal at the refracting surface it encounters. I_0 would be the angle between the original ray and the normal to the surface it originated from. So I_0 is at the beginning and I_1 at the end of the ray of length D.

k, l, and m are the direction cosines of the normal to the surface at which the refraction is taking place. They may be found as follows:

k = 1 - (X' / r) l = -Y' / r m = -Z' / r

Kingslake says, " It is a well-known property of direction cosines that the angle between two intersecting lines is given by: "

$\cos I_1 = Kk + Ll + Mm$

From which he concludes without derivation that:

$\cos I_1 = (F - D) / r$ (Equa. 6)

NB: This is my F, not Kingslake's.

For a planar refracting surface, he gives:

$\cos I_1 = K$ (Equa. 6p)

which is obvious if you think about it.

Part b - Refraction

First you find the cosine of the angle, I_0', that the refracted ray makes to the normal to the refracting surface at the point of intersection:

$$\cos I_0' = [1-(n/n')^2 (1-\cos^2 I_1)]^{1/2} \qquad \text{(Equa. 7)}$$

Then you find an intermediate value called " J' " which you use in subsequent calculations:

$$J' = (n' \cos I_0' - n \cos I_1) / r \qquad \text{(Equa. 8)}$$

Or, if the refracting surface is a plane:

$$J' = 0 \qquad \text{since } r = 0 \qquad \text{(Equa. 8p)}$$

Kingslake gives the refraction equations as:

$n'K' = nK + (n' \cos I_0' - n \cos I_1) k$

$n'L' = nL + (n' \cos I_0' - n \cos I_1) l$

$n'M' = nM + (n' \cos I_0' - n \cos I_1) m$

If you substitute into these equations the definition of J' (Equa. 8 or 8p) and the definitions of the direction cosines of the normal (k, l, m, above), you can find the direction cosines of the refracted ray from the new equations:

$$K' = [nK - J' (X'-r)] / n'$$
$$L' = (nL - J'Y') / n' \qquad \text{(Equa's. 9)}$$
$$M' = (nM - J'Z') / n'$$

Or, for a plane:

$$K' = \cos I_0'$$
$$L' = L(n/n') \qquad \text{(Equa's. 9p)}$$
$$M' = M(n/n')$$

A good check is based on the sum of the squares of the direction cosines:

$$(K')^2 + (L')^2 + (M')^2 - 1 = 0$$

SECTION 0 - Ray tracing an ocular lens

Seymour Rosin in his article on eyepiece (ocular) design in “Applied Optics & Optical Engineering” Vol. 3, Academic Press, says that eyepieces are designed backwards, as if parallel ray pencils exited the pupil of the eye, went through the eyepiece, and landed on the image being observed.

My eye ray conventions; figuring the origin and direction cosines of a ray emanating from the entrance pupil of the eye.

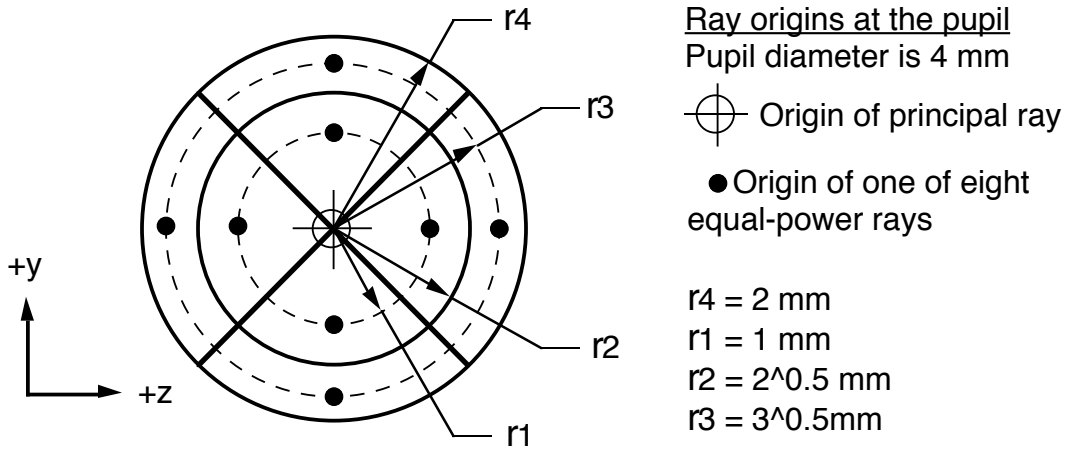


FIGURE 4

The area of the eye's pupil has been broken up into eight equal parts. Each black dot represents the origin of a ray representing a one-eighth region.

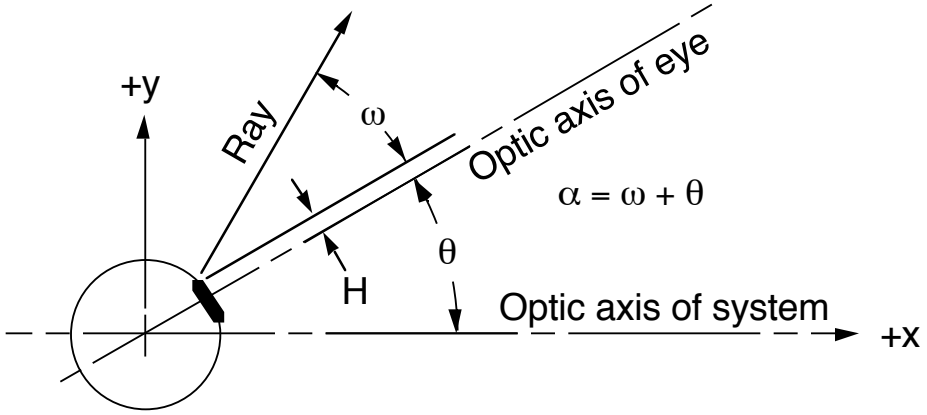


FIGURE 5

There is no H needed in the z direction; I just use the Z input. This can be done because θ in the z direction can be kept at zero with no loss of generality. The ω in the z direction is called β.

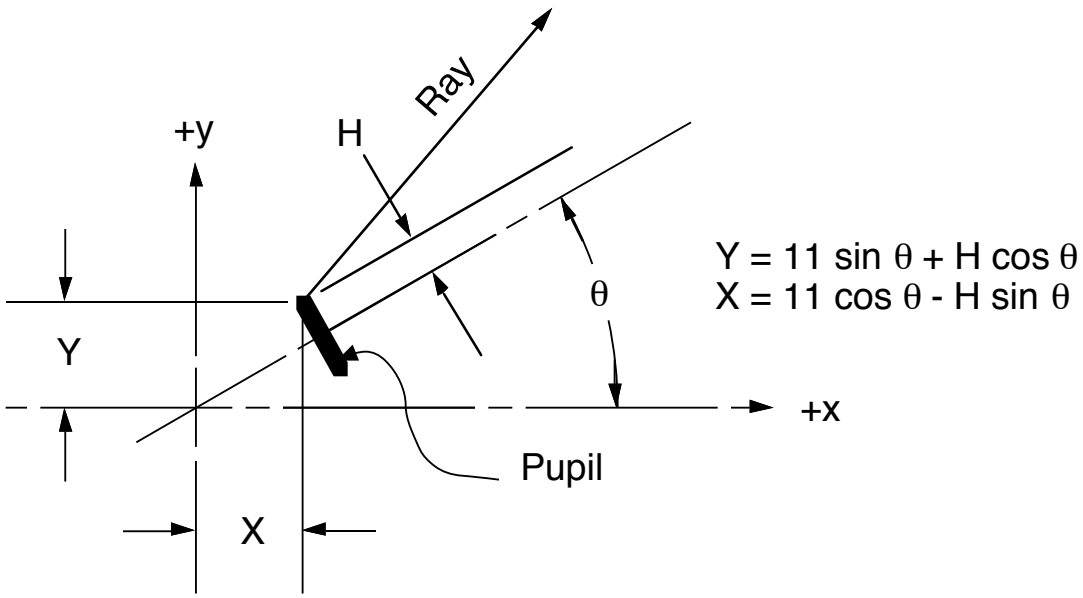


FIGURE 6 (enlargement of Figure 5)

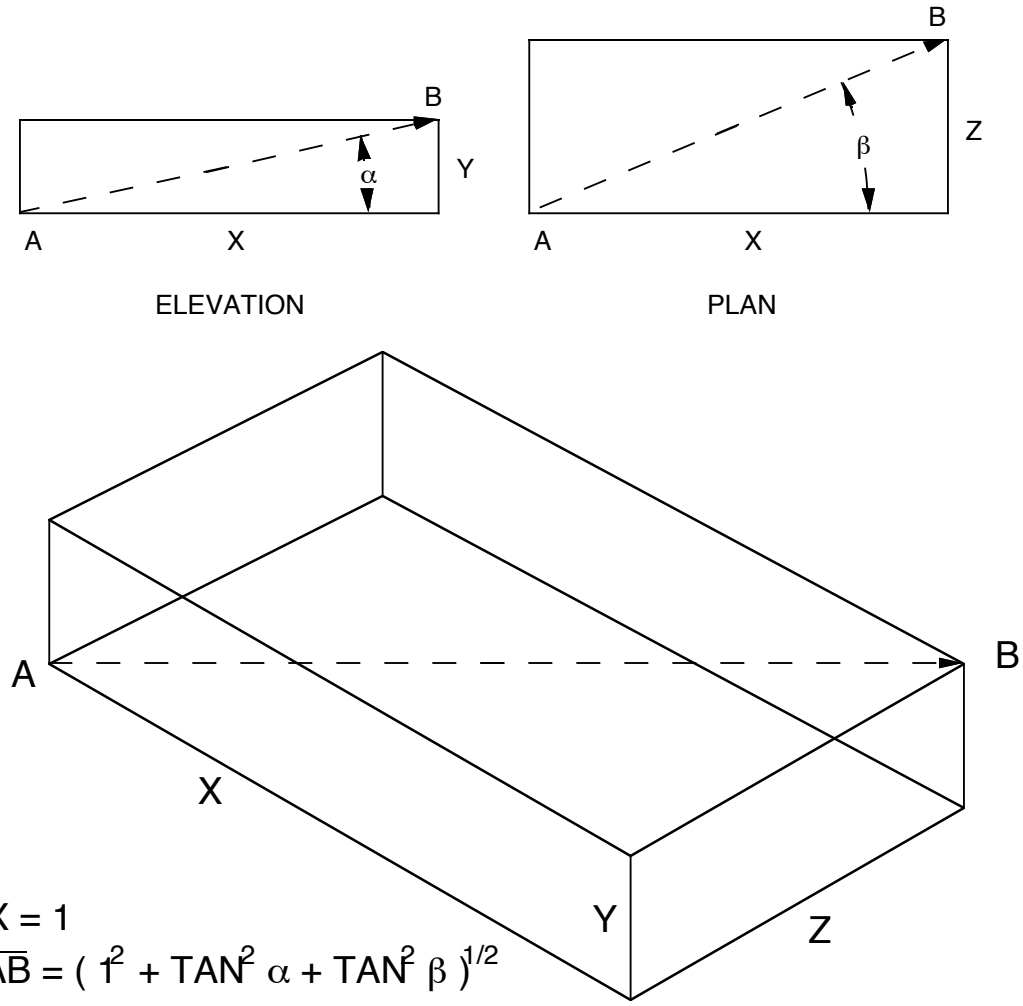


FIGURE 7

Direction cosine K = x /AB = 1/AB
Direction cosine L = y /AB = tan α /AB = K *tan α
Direction cosine M = z /AB = tan β /AB = K *tan β

Point A is the point of origin on the eye's pupil of the ray AB.