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EQUATIONS OF MOTION FOR THE VARIABLE MASS FLOW—VARIABLE EXHAUST VELOCITY ROCKET

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EQUATIONS OF MOTION FOR THE VARIABLE MASS FLOW-VARIABLE EXHAUST VELOCITY ROCKET

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Abstract

An equation of motion for a one dimensional rocket is derived as a function of f(t), the mass flow rate into the acceleration chamber and v(x,t), the velocity distribution along the chamber, thereby including the transient flow changes in the chamber which have not been explicitly contained in other papers. The derivation of the mass density requires the introduction of the special time coordinate τ, which satisfies the equation $0=1-\partial\tau/\partial t-v\partial\tau/\partial x$. The mass density expression is $f(t-\tau)\partial\tau/\partial x$. The equation of motion is derived from both classical force and momentum approaches and is shown to be consistent with the standard equation expressed in terms of flow parameters at the exit to the acceleration chamber.

Introduction

As propulsion systems become more advanced and sophisticated there is a tendency toward exercising more exact control over the mass flow rate into the engine and the acceleration imparted to the mass as it flows through the engine. Hence, the derivation of rocket equations of motion for advanced propulsion systems will have to be based on a rocket model that allows for the mass flow rate into the engine and the velocity distribution along the engine to vary with time.

The objective of the analysis undertaken herein is the acceleration of the rocket expressed in the form:

$$du/dt = F[M, x_s, x_e(t), v(x, t), f(t), x_{cp}(t)] \tag{1.1}$$

where:

M — Initial mass of the rocket.
x_s — Location of the inlet of the acceleration chamber.
$x_e(t)$ — Location of the end of the acceleration chamber.
$v(x,t)$ — Velocity of the mass in the acceleration chamber with respect to the chamber.
f(t) — Mass flow rate into the acceleration chamber.
$x_{cp}(t)$ — Location of the center of mass of the rocket proper (the rocket minus the acceleration chamber).

The above quantities completely define the rocket herein considered and must be initially specified. The acceleration is derived for the time period before the acceleration chamber is full as well as for the period after mass has stopped entering the chamber. The extensive literature[1-11] which considers the variable mass flow-variable exhaust velocity rocket does not contain an equation of motion in the form of Eq. (1.1), which would be the parameters normally specified. The approach taken in the literature is to treat the rocket as one entity, without separating the mass in the acceleration chamber from the rocket proper.

The equations of motion which apply to a variable mass system have been a subject of debate among physicists, resulting in no less than eight articles[12-19] published in the American Journal of Physics during the period 1964-1966 alone. The author finds the following two approaches to be equivalent and applicable to the variable mass problem considered herein.

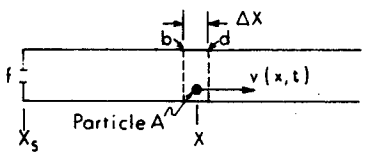
1. The "momentum" approach: The time rate of change of momentum of all the particles involved must equal zero.
2. The "force" approach: The sum of the forces acting on all the particles involved must equal zero.

II. Derivation of the Mass Density in the Acceleration Chamber

An analysis of the flow in the acceleration chamber (see Fig. 1) is required before the equation of motion can be obtained in the form of Eq. (1.1). The acceleration chamber in the rocket is used to accelerate the mass inserted into the chamber to the velocity at which it leaves the chamber. It is assumed that the mass enters the chamber with zero relative velocity. The velocity distribution along the chamber is

$$dx/dt = v = v(x, t) \tag{2.1}$$

where v is a specified function of both distance along the chamber and time. The acceleration of the mass within the chamber can be obtained from this velocity distribution.



τ = Time for particle A to move from x_s to x

$\Delta\tau$ = Difference in times that particles at b and d entered chamber

Figure 1. Acceleration chamber notation

The amount of time τ that the particle at point x_0 at time t_0 has been in the acceleration chamber can be found by numerically integrating the equation

$$dt/dx = 1/v \tag{2.2}$$

between the limits of $x=x_0$, $t=t_0$ and $x=x_s$, $t=t_0-\tau$. τ is then obtained from

$$\tau = t_0 - t = \tau(x_0, t_0) \tag{2.3}$$

If the variables in Eq.(2.1) are separable, the following equation defines τ

$$\int_{t_0-\tau}^{t_0} a\,dt = \int_{x_s}^{x_0} \frac{dx}{b} \tag{2.4}$$

where a and b are defined by

$$v = ab = a(t)\,b(x) \tag{2.5}$$

As x_0 and t_0 are completely arbitrary, the general expression for τ is

$$\tau = \tau(x, t) \tag{2.6}$$

The mass flow rate into the acceleration chamber is given by f, a specified function of time. This function, along with the velocity distribution, determines the mass density in the chamber. Consider the mass Δm that occupies a small length Δx centered at x. An expression for this mass is obtained by multiplying the mass flow at the time that the particle at x left the inlet of the chamber by an interval of time $\Delta\tau$.

$$\Delta m = f^* \Delta\tau \tag{2.7}$$

The functions f and df/dt when starred should be modified by substituting $t-\tau$ for t. For example, if $t=t^3$, $df/dt=3t^2$, then $f^*=(t-\tau)^3$, $(df/dt)^*=3(t-\tau)^2$. The time interval $\Delta\tau$ is the difference in times that the particles on the right and left sides of Δx entered the inlet of the acceleration chamber and is given by (see Fig. 1)

$$\Delta\tau = \frac{\partial\tau}{\partial x}\Delta x \tag{2.8}$$

The mass density ρ is defined as the mass per unit length, yielding

$$\rho = \frac{dm}{dx} = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x} = f^* \frac{\partial\tau}{\partial x} \tag{2.9}$$

Equation (2.9) may be demonstrated to be valid by using the equation of continuity

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0 \tag{2.10}$$

Differentiating Eq. (2.9):

$$\frac{\partial\rho}{\partial t} = f^* \frac{\partial^2\tau}{\partial x\,\partial t} + \left(1 - \frac{\partial\tau}{\partial t}\right)\frac{\partial\tau}{\partial x}\left(\frac{df}{dt}\right)^*$$

$$\frac{\partial\rho}{\partial x} = f^* \frac{\partial^2\tau}{\partial x^2} - \left(\frac{\partial\tau}{\partial x}\right)^2 \left(\frac{df}{dt}\right)^* \tag{2.11}$$

Inserting Eq. (2.11) into the equation of continuity gives

$$0 = \left(1 - \frac{\partial\tau}{\partial t} - v\frac{\partial\tau}{\partial x}\right)\frac{\partial\tau}{\partial x}\left(\frac{df}{dt}\right)^* + \left(\frac{\partial^2\tau}{\partial x\,\partial t} + \frac{\partial v}{\partial x}\frac{\partial\tau}{\partial x} + v\frac{\partial^2\tau}{\partial x^2}\right) f^* \tag{2.12}$$

The easiest way to show that Eq.(2.12) is valid is to introduce a variable T.

$$T = t - \tau \tag{2.13}$$

T is assumed to be associated with the particle that occupies x at time t. Partial differentiation of T results in

$$\frac{\partial T}{\partial x} = -\frac{\partial\tau}{\partial x}, \quad \frac{\partial T}{\partial t} = 1 - \frac{\partial\tau}{\partial t} \tag{2.14}$$

As time elapses, this particle moves down the chamber. τ increases uniformly with t. Hence, the total differential of T equals zero.

$$dT = 0 = \frac{\partial T}{\partial x}\,dx + \frac{\partial T}{\partial t}\,dt \tag{2.15}$$

Substituting Eq. (2.14) into Eq. (2.15) gives

$$0 = 1 - \frac{\partial\tau}{\partial t} - v\frac{\partial\tau}{\partial x} \tag{2.16}$$

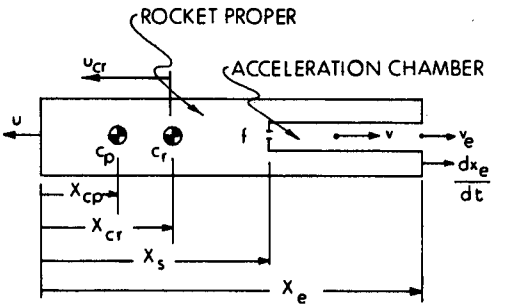
As Eq. (2.13) was for an arbitrary x, the partial differential of Eq. (2.16) with respect to x must equal zero.

$$0 = \frac{\partial^2\tau}{\partial x\,\partial t} + \frac{\partial v}{\partial x}\frac{\partial\tau}{\partial x} + v\frac{\partial^2\tau}{\partial x^2} \tag{2.17}$$

Considering Eq. (2.16) and Eq. (2.17) it is apparent that Eq. (2.12) is valid.

III. Derivation of Equation of Motion When Acceleration Chamber is Full

This section contains a derivation of the equation of motion for the variable mass flow-variable exhaust velocity rocket for the time period when the chamber is full. Figure 2 shows the rocket nomenclature. All distances measured relative to the rocket are measured from a frontal position so that an x coordinate will always be positive in a rearward direction.



c_p - Position of center of mass of the rocket proper
c_r - Position of center of mass of the whole rocket
u, u_{cr} - Absolute velocities
v, v_e - Velocities relative to rocket

Figure 2. Rocket nomenclature

The mass of the rocket m_r consists of the mass in the rocket proper plus the mass in the acceleration chamber

$$m_r = m_p + \int_{x_s}^{x_e} \rho\,dx \tag{3.1}$$

where x_s and x_e denote the start and end of the acceleration chamber. The distance x_e is assumed to vary with time. Differentiating Eq. (3.1) using Leibniz's rule[20] and considering x_s to be constant results in

$$\frac{dm_r}{dt} = \frac{dm_p}{dt} + \rho_e\frac{dx_e}{dt} + \int_{x_s}^{x_e} \frac{\partial\rho}{\partial t}\,dx \tag{3.2}$$

where ρ_e is the density at x_e. Using the equation of continuity (Eq. (2.10))

$$\int_{x_s}^{x_e} \frac{\partial \rho}{\partial t}\, dx = -\rho_e v_e - \frac{dm_p}{dt} \tag{3.3}$$

where v_e is the velocity of the mass in the acceleration chamber at x_e and $\rho_s v_s$ has been set equal to $-dm_p/dt$. Combining Eq. (3.2) and Eq. (3.3) gives the mass flow rate leaving the rocket

$$\frac{dm_r}{dt} = \rho_e \left(\frac{dx_e}{dt} - v_e\right) \tag{3.4}$$

The momentum p of the system (ie: the momentum of all the particles that were initially in the rocket) can be expressed as the sum of the momentum of the particles in the rocket proper plus the particles in the acceleration chamber plus the particles that have already left the rocket. The momentum shall be expressed in an inertial coordinate system which was at rest with respect to the rocket before it started to move.

The rocket proper, denoted with subscript p, consists of a group of particles which are loosing mass at point x_s. The center of mass x_c of the rocket proper varies with time and is assumed to be specified. Appendix A contains a derivation (Eq.(A.9)) of the momentum of this body

$$p_p = u m_p - \frac{dm_p\, x_{cp}}{dt} + x_s \frac{dm_p}{dt} \tag{3.5}$$

where u is the velocity of a point fixed to the acceleration chamber of the rocket. Using the mass density expression (Eq.(2.9)), the momentum of the mass in the acceleration chamber is

$$p_{ac} = \int_{x_s}^{x_e} \rho\, (u - v)\, dx \tag{3.6}$$

Using the mass flow rate (Eq. (3.4)) out of the rocket, the momentum p_e of the mass which has left the rocket can be expressed as

$$p_e = \int_{t_s}^{t} \rho_e \left(v_e - \frac{dx_e}{dt}\right)(u - v_e)\, dt \tag{3.7}$$

where t_s is the time at which mass first starts to leave the rocket. Combining Eqs. (3.5), (3.6) and (3.7) results in the total momentum of the original mass of the rocket.

$$p = u m_p - \frac{dm_p\, x_{cp}}{dt} + x_s \frac{dm_p}{dt} + \int_{x_s}^{x_e} \rho\, (u - v)\, dx + \int_{t_s}^{t} \rho_e \left(v_e - \frac{dx_e}{dt}\right)(u - v_e)\, dt \tag{3.8}$$

As the total momentum of the system is conserved, Eq. (3.8) may be differentiated and set equal to zero. The result is

$$0 = m_p \frac{du}{dt} + u \frac{dm_p}{dt} - \frac{d^2 m_p\, x_{cp}}{dt^2} + x_s \frac{d^2 m_p}{dt^2} + \rho_e (u - v_e) \frac{dx_e}{dt} + \rho_e \left(v_e - \frac{dx_e}{dt}\right)(u - v_e) + \frac{du}{dt} \int_{x_s}^{x_e} \rho\, dx + u \int_{x_s}^{x_e} \frac{\partial \rho}{\partial t}\, dx - \int_{x_s}^{x_e} \frac{\partial \rho v}{\partial t}\, dx \tag{3.9}$$

Simplifying the above equation using Eqs. (3.1) and (3.3) gives

$$0 = m_r \frac{du}{dt} - \frac{d^2 m_p\, x_{cp}}{dt^2} + x_s \frac{d^2 m_p}{dt^2} - \rho_e v_e^2 - \int_{x_s}^{x_e} \frac{\partial \rho v}{\partial t}\, dx \tag{3.10}$$

The mass of the rocket equals the mass of the rocket at time t_s minus what has left the rocket

$$m_r = M - \int_{t_s}^{t} \rho_e \left(v_e - \frac{dx_e}{dt}\right) dt \tag{3.11}$$

The mass of the rocket proper equals

$$m_p = M - \int_{0}^{t} f\, dt \tag{3.12}$$

The following equation reduces to the continuity equation when expanded

$$\frac{\partial \rho v}{\partial t} = \rho \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x}\right) - \frac{\partial \rho v^2}{\partial x} = \rho \frac{dv}{dt} - \frac{\partial \rho v^2}{\partial x} \tag{3.13}$$

Combining Eqs. (3.10)-(3.13) and (2.9) yields the equation of motion for the variable mass flow-variable exhaust velocity rocket

$$0 = \left[M - \int_{t_s}^{t} f_e^* \left(\frac{\partial \tau}{\partial x}\right)_e \left(v_e - \frac{dx_e}{dt}\right) dt\right] \frac{du}{dt} + (x_{cp} - x_s) \frac{df}{dt} + 2f \frac{dx_{cp}}{dt} - \left(M - \int_{0}^{t} f\, dt\right) \frac{d^2 x_{cp}}{dt} - \int_{x_s}^{x_e} f^* \frac{\partial \tau}{\partial x} \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x}\right) dx \tag{3.14}$$

where f_e^* and $(\partial \tau / \partial x)_e$ are computed at $x = x_e$. This is the nearest one can come to deriving a general expression in the form of Eq. (1.1). The only quantity in the above equation which is not contained in Eq. (1.1) is τ, which can be explicitly obtained only for certain functions of v (see Section VIII for an example).

The above analysis is consistent with the results obtained by those who treat the variable mass flow-variable exhaust velocity rocket as a variable mass problem in which the center of mass of the whole rocket plays the leading role. The main result of these approaches is equation B-10, which can be expressed in terms of the nomenclature of Fig. 2.

$$0 = m_r \frac{du_{cr}}{dt} + (x_e - x_{cr}) \frac{d^2 m_r}{dt^2} + \left(v_e + \frac{dx_e}{dt} - 2 \frac{dx_{cr}}{dt}\right) \frac{dm_r}{dt} \tag{3.15}$$

where $m_r = m$, $u_{cr} = u_c$, $x_{cr} = x_c$, $x_e = x_1$, and $v_e = v_1$. du_{cr}/dt is the acceleration of the center of mass of the rocket in an inertial coordinate system and x_{cr} is the position of the center of mass of the rocket (see Fig. 2).

It will now be shown that Eq. (3.10) is equivalent to Eq. (3.15). The center of mass of the rocket is obtained from

$$m_r x_{cr} = m_p x_{cp} + \int_{x_s}^{x_e} x\rho\, dx \tag{3.16}$$

Differentiating results in

$$\frac{dm_r x_{cr}}{dt} = \frac{dm_p x_{cp}}{dt} + x_e \rho_e \frac{dx_e}{dt} + \int_{x_s}^{x_e} x \frac{\partial \rho}{\partial t}\, dx \tag{3.17}$$

The following identity can be established by integration by parts using the continuity equation.

$$\int_{x_1}^{x_2} x \frac{\partial \rho}{\partial t}\, dx = x_1 \rho_1 v_1 - x_2 \rho_2 v_2 + \int_{x_1}^{x_2} \rho v\, dx \tag{3.18}$$

Combining Eqs. (3.4), (3.17) and (3.18) and setting $dm_p/dt = -\rho_s v_s$ gives

$$\frac{dm_r x_{cr}}{dt} = \frac{dm_p x_{cp}}{dt} + x_e \frac{dm_r}{dt} - x_s \frac{dm_p}{dt} + \int_{x_s}^{x_e} v\, dx \tag{3.19}$$

Differentiating the above equation results in

$$\frac{d^2 m_r x_{cr}}{dt^2} = \frac{d^2 m_p x_{cp}}{dt^2} + \frac{dx_e}{dt}\frac{dm_r}{dt} - x_s \frac{d^2 m_p}{dt^2} + x_e \frac{d^2 m_r}{dt^2} + \rho_e v_e \frac{dx_e}{dt} + \int_{x_s}^{x_e} \frac{\partial \rho v}{\partial t}\, dx \tag{3.20}$$

Combining Eqs. (3.4), (3.10) and (3.20) gives

$$0 = m_r \frac{du}{dt} - m_r \frac{d^2 x_{cr}}{dt^2} + (x_e - x_{cr}) \frac{d^2 m_r}{dt^2} + \left(v_e + \frac{dx_e}{dt} - 2\frac{dx_{cr}}{dt}\right) \frac{dm_r}{dt} \tag{3.21}$$

The inertial acceleration (du_{cr}/dt) of the center of mass of the rocket is

$$\frac{du_{cr}}{dt} = \frac{du}{dt} - \frac{d^2 x_{cr}}{dt^2} \tag{3.22}$$

Combining Eqs. (3.21) and (3.22) yields the desired result, Eq. (3.15).

Equation (3.10) is also consistent with the approach taken in references 5 and 6. Miele applies Newton's Second Law of Motion to the rocket regarded as a continuum in the following manner.

$$F = \int_m \left(\frac{du}{dt} - \frac{dv}{dt}\right) dm = m_r \frac{du}{dt} - \int_{m_r} \frac{dv}{dt}\, dm \tag{3.23}$$

where F is the external force, du/dt is the acceleration of the origin (point fixed on the rocket), dv/dt is the relative acceleration with respect to the origin and the integral is to be summed over the mass of the rocket. Although Miele extends his approach further, it is easiest to derive Eq. (3.10) directly from Eq. (3.23).

Setting the external force equal to zero and separating the integral into an integral of the mass in the rocket proper and an integral of the mass in the acceleration chamber results in

$$0 = m_r \frac{du}{dt} - \int_{x_s}^{x_e} \rho \frac{dv}{dt}\, dx - \int_0^{x_s} \rho_p \frac{dv}{dt}\, dx - \int_{x_s}^{x_e} \rho_p \frac{dv}{dt}\, dx \tag{3.24}$$

in terms of the notation in Fig. 1 and where ρ_p is the density in the rocket proper (for $0<x<x_e$). Appendix A contains a derivation of the equation

$$\frac{dm_p x_{cp}}{dt} = x_s \frac{dm_p}{dt} + \int_0^{x_s} \rho_p v\, dx + \int_{x_s}^{x_e} \rho_p v\, dx \tag{3.25}$$

Differentiating, setting $\rho_{pe} v_e\, dx_e/dt = 0$, and using Eq. (3.13) gives

$$\frac{d^2 m_p x_{cp}}{dt^2} = x_s \frac{d^2 m_p}{dt^2} + \int_0^{x_s} \rho_p \frac{dv}{dt}\, dx + \int_{x_s}^{x_e} \rho_p \frac{dv}{dt}\, dx \tag{3.26}$$

where it has been assumed that $\rho_p v^2$ is equal to zero at x equal to 0, x_s and x_e. Equation (3.24) can be obtained by combining Eqs. (3.10), (3.13), and (3.26).

IV. Derivation of Equation of Motion Before Mass Leaves the Acceleration Chamber

The equation of motion for the time interval before the mass starts to leave from the acceleration chamber is found in a similar manner as was the main equation of motion (Eq.(3.14)). Themomentum expression can be obtained from Eq. (3.8) after eliminating the term representing the expelled mass

$$p = u m_p - \frac{dm_p x_{cp}}{dt} + x_s \frac{dm_p}{dt} + \int_{x_s}^{x_1} \rho (u - v)\, dx \tag{4.1}$$

where x_1 is the point reached by the first particle that entered the acceleration chamber. X_1 is obtained by integrating Eq.(2.1). The mass of the rocket proper is given by Eq.(3.12). Differentiating Eq.(4.1), setting equal to zero and using Eq.(2.10) and the derivative of Eq.(3.12) results in

$$0 = M \frac{du}{dt} - \frac{d^2 m_p x_{cp}}{dt^2} - x_s \frac{df}{dt} - \rho_1 v_1^2 - \int_{x_s}^{x_1} \frac{\partial \rho v}{\partial t}\, dx \tag{4.2}$$

where v_1 is the velocity at x_1. Combining Eqs.(3.12) (3.13) and (4.2) gives

$$0 = M \frac{du}{dt} + (x_{cp} - x_s) \frac{df}{dt} + 2f \frac{dx_{cp}}{dt} - \left(M - \int_0^t f dt\right) \frac{d^2 x_{cp}}{dt^2} - \int_{x_s}^{x_1} f^* \frac{\partial \tau}{\partial x} \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x}\right) dx \tag{4.3}$$

The only difference between Eq. (3.14) and the above equation besides the obvious ones of interchanging x_1 for x_e and the dropping of the integral in the first term is that in the above equation the derivative of x_1 with respect to time is v_1, whereas in Eq. (3.14) x_e is a specified function of time.

V. Derivation of Equation of Motion After Mass Stops Entering The Acceleration Chamber

Once the mass stops entering the acceleration chamber, the mass remaining in the rocket proper remains constant. If t_1 represents the time that the last particle entered the chamber, at time t it would have reached point x_2 found by integrating Eq. (2.1). The integral representing the momentum in the chamber need cover only the range from x_2 to x_e. Equation (3.8) for this case becomes

$$p = m_p \left(u - \frac{dx_{cp}}{dt}\right) + \int_{x_2}^{x_e} \rho\,(u - v)\,dx + \int_{t_s}^{t} \rho_e \left(v_e - \frac{dx_e}{dt}\right)(u - v_e)\,dt \tag{5.1}$$

The mass of the rocket can be expressed

$$m_r = m_p + \int_{x_2}^{x_e} \rho\,dx \tag{5.2}$$

Differentiating Eq. (5.1), setting equal to zero and using Eqs. (2.10) and (5.2) results in

$$0 = m_r \frac{du}{dt} - m_p \frac{d^2x_{cp}}{dt^2} + \rho_2 v_2^2 - \rho_e v_e^2 - \int_{x_2}^{x_e} \frac{\partial \rho v}{\partial t}\,dx \tag{5.3}$$

Combining Eqs. (3.11), (3.12), (3.13) and (5.3) gives

$$0 = \left[M - \int_{t_s}^{t} f_e^* \left(\frac{\partial \tau}{\partial x}\right)_e \left(v_e - \frac{dx_e}{dt}\right) dt\right] \frac{du}{dt} - \left(M - \int_0^{t_1} f\,dt\right) \frac{d^2x_{cp}}{dt^2} - \int_{x_2}^{x_e} f^* \frac{\partial \tau}{\partial x} \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x}\right) dx \tag{5.4}$$

The differences between Eqs. (3.14) and (5.4) are the changes in the limits on the integrals and the dropping of the terms containing x_s and f.

VI. General Equation Of Motion

By comparing the equations of motion (Eqs. (3.14), (4.4) and (5.4)) for the different time periods considered in the previous three sections, the following general equation of motion for the variable mass flow-variable exhaust velocity rocket can be obtained

$$0 = m_r \frac{du}{dt} + (x_{cp} - x_s) \frac{df}{dt} + 2f \frac{dx_{cp}}{dt} - \left(M - \int_0^{t_a} f\,dt\right) \frac{d^2x_{cp}}{dt^2} - \int_{x_a}^{x_b} f^* \frac{\partial \tau}{\partial x} \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x}\right) dx \tag{6.1}$$

where: $m_r = M$ if no mass has left the rocket. Otherwise

$$m_r = M - \int_{t_s}^{t} f_e^* \left(\frac{\partial \tau}{\partial x}\right)_e \left(v_e - \frac{dx_e}{dt}\right) dt \tag{6.2}$$

x_a equals x_s if mass is currently entering the acceleration chamber. Otherwise it is the point occupied by the mass in the chamber closest to the inlet.

x_b equals x_e if mass is currently leaving the acceleration chamber. Otherwise it is the point occupied by the mass in the chamber closest to the exit.

t_a equals t if $t < t_1$; otherwise $t_a = t_1$.

The points x_a and x_b when not equal to x_s or x_e respectively, are functions of time; they move with a velocity equal to the velocity in the acceleration chamber at that point.

VII. Special Cases

Several rocket models lead to simplified density expressions and corresponding simplified equations of motion. In the following examples both x_{cp} and x_e are assumed to be independent of time and x_s is assume to occur at x_{cp}. In addition, the reference point in the rocket is assumed to coincide with x_s. Attention is restricted to the full acceleration chamber situation.

A. Exhaust Velocity that Allows Separation of Variables

Assume that v is given by

$$v = ab = a(t)\,b(x) \tag{7.1}$$

Differentiating Eq. (2.4) and using Eq. (2.9) gives

$$\partial\tau/\partial x = 1/a^*b, \quad \rho = f^*/a^*b \tag{7.2}$$

Equation (3.14) reduces to

$$0 = \left[M - \int_{t_s}^{t} \frac{a f_e^*}{a^*}\,dt\right] \frac{du}{dt} - \int_0^{x_e} \frac{f^*}{a^*} \left(\frac{da}{dt} + a^2 \frac{db}{dx}\right) dx \tag{7.3}$$

Assuming further that f=a, the density $\rho = 1/b$ and Eq. (7.3) reduces to

$$0 = \left(M - \int_{t_s}^{t} a\,dt\right) \frac{du}{dt} - x_e \frac{da}{dt} - a^2 b_{x_e} \tag{7.4}$$

B. Constant Exhaust Velocity Rocket

The constant exhaust velocity rocket implies that the exhaust velocity does not change with time while the mass flow is variable. Hence,

$$v = b = b(x) \tag{7.5}$$

Differentiating Eq. (2.4) with a equal to one gives

$$\partial\tau/\partial x = 1/b \tag{7.6}$$

Eqs. (2.9) and (3.14) reduce to

$$\rho = f^*/b$$

$$0 = (M - \int_{t_s}^{t} f_e^* \, dt) \frac{du}{dt} - \int_0^{x_e} f^* \frac{\partial v}{\partial x} \, dx \qquad (7.7)$$

where $\tau = \int_0^x \frac{dx}{b}$

C. Constant Mass Flow Rate Rocket

The constant mass flow rate rocket implies that the mass flow into the acceleration chamber does not change with time while the exhaust velocity does change with time. Hence,

$$f = c \qquad (7.8)$$

where c is a constant. Eqs. (2.9) and (3.14) reduce to

$$\rho = c \, \partial\tau/\partial x$$

$$0 = \left[M - c \int_{t_s}^{t} (\frac{\partial \tau}{\partial x})_e \, v_e \, dt \right] \frac{du}{dt} - c \int_{x_s}^{x_e} \frac{\partial \tau}{\partial x} (\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x}) \, dx \qquad (7.9)$$

D. Steady State Rocket

Both the mass flow rate and exhaust velocity are independent of time for the steady state rocket. If

$$f = c, \; v = b = b(x) \qquad (7.10)$$

then Eqs. (2.9) and (3.14) reduce to

$$\rho = c/b$$

$$0 = \left[M - c(t - t_s) \right] \frac{du}{dt} - c \, b_e \qquad (7.11)$$

where b_e is based on $x = x_e$. Integration of Tsiolkovskii's equation gives

$$u - u_s = b_e \log \left\{ M / [M - c(t - t_s)] \right\} \qquad (7.12)$$

where u_s is the velocity at time t_s.

VIII. Numerical Example

This section contains an example of when the variables in v are separable and such that Eq. (2.4) can be integrated. The same assumptions concerning x_{cp} and x_e are made as was assumed in Section VII. Let the mass flow rate into the acceleration chamber and the velocity be specified by

$$f = t, \; v = tx^n \qquad (8.1)$$

The acceleration of the mass in the acceleration chamber is given by

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = x^n + nt^2 \, x^{2n-1} \qquad (8.2)$$

Carrying out the integration of Eq. (2.4) results in τ

$$\tau = t - \left[t^2 - 2x^{1-n}/(1-n) \right]^{1/2} \qquad (8.3)$$

Eliminating x between Eqs. (8.2) and (8.3) gives

$$\frac{dv}{dt} = \left\{ \frac{1-n}{2} \left[t^2 - (t - \tau)^2 \right] \right\}^{\frac{n}{1-n}} + nt^2 \left\{ \frac{1-n}{2} \left[t^2 - (t - \tau)^2 \right] \right\}^{\frac{2n-1}{1-n}} \qquad (8.4)$$

Hence, to avoid infinite accelerations at the inlet (i.e.: when τ goes to zero), n must lie in the range $1/2 < n < 1$.

Differentiating Eq. (8.3) yields

$$\frac{\partial \tau}{\partial x} = x^{-n} (t^2 - \frac{2}{1-n} x^{1-n})^{-1/2} \qquad (8.5)$$

The density (Eq. (2.9) and mass flow rate are

$$\rho = 1/x^n; \; \rho v = t \qquad (8.6)$$

Equation (4.4) reduces to

$$0 = M \frac{du}{dt} - x_1 - x_1^n \, t^2 \qquad (8.7)$$

Integrating the velocity to get the time-position history of the first particle that enters the chamber results in

$$x_1 = \left[(1-n) \, t^2/2 \right]^{1/2} \qquad (8.8)$$

Inserting into Eq. (8.7) and integrating gives

$$u = \frac{1}{M} \frac{1-n}{3-n} t^{\frac{3-n}{1-n}} \left[(\frac{1-n}{2})^{\frac{n}{1-n}} + (\frac{1-n}{2})^{\frac{1}{1-n}} \right] \qquad (8.9)$$

This equation is valid until the chamber is full at time t_s. This time is found by solving Eq. (8.8) for t after substituting x_e for x_1.

$$t_s = \left[2 x_e^{1-n} / (1-n) \right]^{1/2} \qquad (8.10)$$

For times in excess of t_s, Eq. (3.14) reduces to

$$0 = (M - t^2/2 + t_s^2/2) \frac{du}{dt} - x_e - x_e^n \, t^2 \qquad (8.11)$$

Integrating

$$u - u_s = x_e \, (a - b)/c + 2x_e^n \, (-t + ac/2 + t_s - bc/2) \qquad (8.12)$$

where

$$c = (2M + t_s^2)^{1/2}, \; a = \log \left[(c + t)/(c - t) \right]$$

$$b = \log \left[(c + t_s)/(c - t_s) \right]$$

This equation is valid until mass stops entering the chamber at time t_1. For greater times the equation of motion is found using Eq.(5.4)

$$0 = (M - t^2/2 + t_s^2/2)\frac{du}{dt} - x_e^n t^2 - x_e + x_2^n t^2 + x_2 \qquad (8.13)$$

where x_2 is obtained by integrating the velocity.

$$x_2 = \left[(1-n)(t^2 - t_1^2)/2\right]^{1/(1-n)} \qquad (8.14)$$

As Eq. (8.13) is not readily integrable, it can be solved numerically on a computer. Figure 3 is a plot of u verses t for the following conditions

$$v = x^n t,\ f = t,\ n = .6,\ M = 1$$
$$m_p \text{ at } t_s = .75,\ m_p \text{ at } t_1 = .25 \qquad (8.15)$$

The following times are then derived from Eq. (3.12): $t_s=\sqrt{.5}$, $t_1=\sqrt{1.5}$.

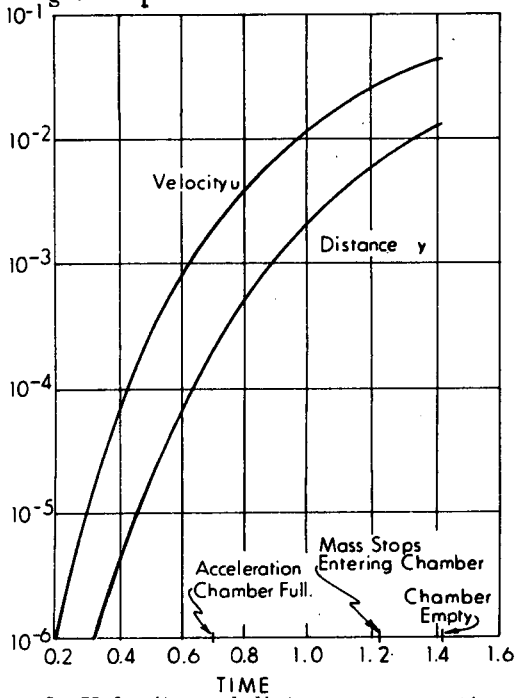


Figure 3. Velocity and distance versus time for rocket defined in Eq. (8.15)

The length of the acceleration chamber equals .00316 and is found by setting x_1 and t equal to x_e and t_s in Eq. (8.8). The time that the last particle leaves the acceleration chamber equals $\sqrt{2}$ and is determined by setting x_2 and t equal to x_e and t_e in Eq. (8.14). The distance the rocket has moved is found by numerically integrating the equation dy/dt = u and is also shown in Fig. 3.

IX. Acknowledgment

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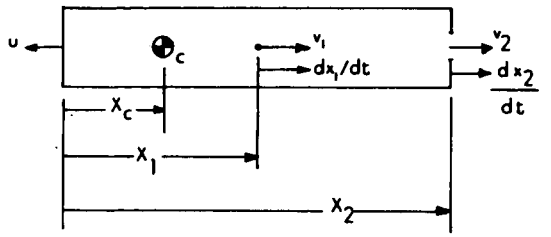
Appendix A
Deprivation of the Momentum of A Variable Mass Body In Terms Of Its Center Of Mass

The derivation of the momentum of a variable mass body in terms of its center of mass can be based on a discontinuous medium[2] or on a continuous medium[1,9]. To be consistant with the general approach taken in this paper, a body with a continuous medium will be used. The body is assumed to lose mass at both point x_1 and x_2, with x_1 and x_2 moving with time (see Fig. 4). The mass leaves with the local velocity v of the mass in the medium.

The location x_c of the center of mass is given by

$$m x_c = \int_0^{x_1} x \rho \, dx + \int_{x_1}^{x_2} x \rho \, dx \qquad (A.1)$$



c - Position of center of mass
u - Absolute velocity
v - Velocity relative to body

Figure 4. Variable mass body notation

There is a discontinuity in the mass density at x_1 due to the mass leaving at that point. If ρ_{01} and ρ_{12} refer to the mass densities on the left and right sides of x_1, differentiating Eq. (A.1) results in

$$\frac{dmx_c}{dt} = x_1 \rho_{01} \frac{dx_1}{dt} + x_2 \rho_2 \frac{dx_2}{dt} - x_1 \rho_{12} \frac{dx_1}{dt} + \int_0^{x_1} x \frac{\partial \rho}{\partial t} dx + \int_{x_1}^{x_2} x \frac{\partial \rho}{\partial t} dx \qquad (A.2)$$

Using Eq. (3.18) the above equation can be reduced to

$$\frac{dmx_c}{dt} = x_2 \rho_2 \left(\frac{dx_2}{dt} - v_2\right) + x_1 (\rho_{01} - \rho_{12}) \left(\frac{dx_1}{dt} - v_1\right) + \int_0^{x_1} \rho v \, dx + \int_{x_1}^{x_2} \rho v \, dx \qquad (A.3)$$

The mass in the body equals

$$m = \int_0^{x_1} \rho \, dx + \int_{x_1}^{x_2} \rho \, dx \qquad (A.4)$$

Differentiating and using the continuity equation gives

$$\frac{dm}{dt} = \rho_2 \left(\frac{dx_2}{dt} - v_2\right) + (\rho_{01} - \rho_{12}) \left(\frac{dx_1}{dt} - v_1\right) \qquad (A.5)$$

The first term on the right hand side of the equation represents the mass flow leaving point 2 and the second term the mass flow leaving point 1. Combining Eqs. (A.3) and (A.5) results in

$$m \frac{dx_c}{dt} + \rho_2 \left(\frac{dx_2}{dt} - v_2\right)(x_c - x_2) + \left(\frac{dx_1}{dt} - v_1\right)(\rho_{01} - \rho_{12})(x_2 - x_1) = \int_0^{x_1} \rho v \, dx + \int_{x_1}^{x_2} \rho v \, dx \qquad (A.6)$$

The momentum of the body is defined to be

$$p = \int_0^{x_1} \rho (u - v) \, dx + \int_{x_1}^{x_2} \rho (u - v) \, dx \qquad (A.7)$$

where u is the velocity of a point fixed on the body. Comparing Eqs. (A.6) and (A.7) gives

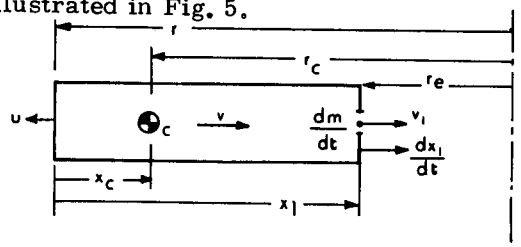
$$p = u \int_0^{x_1} \rho \, dx + u \int_{x_1}^{x_2} \rho \, dx - m \frac{dx_c}{dt} + \rho_2 \left(\frac{dx_2}{dt} - v_2\right)(x_2 - x_c) + \left(\frac{dx_1}{dt} - v_1\right)(\rho_{01} - \rho_{12})(x_1 - x_c) \qquad (A.8)$$

Equations (A.6) and (A.8) can be applied to the rocket proper section of Fig. 1. Using the notation of Fig. 1 and Eq. (A.5) with $\rho_e (dx_2/dt - v_2)$ set equal to zero, Eqs. (A.6) and (A.8) reduce to

$$m \frac{dx_c}{dt} + (x_c - x_s) \frac{dm}{dt} = \int_0^{x_s} \rho v \, dx + \int_{x_s}^{x_e} \rho v \, dx$$
$$p = um - m \frac{dx_c}{dt} + (x_s - x_c) \frac{dm}{dt} \qquad (A.9)$$

Appendix B.
Equation of Motion For A Variable Mass Body In Terms Of Its Center Of Mass

This appendix contains a derivation of an equation which has been frequently mentioned in the literature[1-4] as the equation of motion of a variable mass body (of which the rocket is one example). The derivation follows the approach taken in the main section of this paper (differentiating the momentum of the system and setting the resulting expression equal to zero). The body considered is illustrated in Fig. 5.



c - Position of center of mass
u - Absolute velocity (=dr/dt)
v - Velocity relative to body
dm/dt - Mass flow rate at x_1

Figure 5. Variable mass body notation

The mass within the body is

$$m = \int_0^{x_1} \rho \, dx \qquad (B.1)$$

Assuming that the body loses mass at point x_1 which varies with time, Eq. (B.1) can be differentiated using Eq. (2.10), resulting in

$$\frac{dm}{dt} = \rho_1 \left(\frac{dx_1}{dt} - v_1\right) \qquad (B.2)$$

where v_1 is the velocity of the mass being exhausted.

The momentum of the body plus the momentum of the mass that has left the body equals

$$p = \int_0^{x_1} \rho (u - v) \, dx + \int_{t_s}^{t} \rho_1 \left(v_1 - \frac{dx_1}{dt}\right)(u - v_1) \, dt \qquad (B.3)$$

where u is the velocity of the body and t_s is the time that the body started to lose mass. Differentiating Eq. (B.3) using Eqs. (2.10) and (B.2) gives

$$\frac{dp}{dt} = 0 = m \frac{du}{dt} - \rho_1 v_1^2 - \int_0^{x_1} \frac{\partial \rho v}{\partial t} dx \qquad (B.4)$$

The location x_c of the center of mass is given by

$$mx_c = \int_0^{x_1} x \rho \, dx \qquad (B.5)$$

Differentiating gives

$$\frac{dmx_c}{dt} = x_1 \rho_1 \frac{dx_1}{dt} + \int_0^{x_1} x \frac{\partial \rho}{\partial t} dx \qquad (B.6)$$

where dx_c/dt is the velocity of the center of mass with respect to the body. Using Eqs. (3.18), (B.2) and (B.6) results in

$$\frac{dmx_c}{dt} = x_1 \frac{dm}{dt} + \int_0^{x_1} \rho v \, dx \qquad \text{(B.7)}$$

Differentiating once more gives

$$\frac{d^2mx_c}{dt^2} = x_1 \frac{d^2m}{dt^2} + \left(\frac{dm}{dt} + \rho_1 v_1\right) \frac{dx_1}{dt} + \int_0^{x_1} \frac{\partial \rho v}{\partial t} \, dx \qquad \text{(B.8)}$$

The acceleration of the center of mass point is

$$\frac{du_c}{dt} = \frac{du}{dt} - \frac{d^2x_c}{dt^2} \qquad \text{(B.9)}$$

Combining Eqs. (B.2), (B.4), (B.8) and (B.9) results in

$$0 = m \frac{du_c}{dt} + (x_1 - x_c) \frac{d^2m}{dt^2} + \left(v_1 + \frac{dx_1}{dt} - 2 \frac{dx_c}{dt}\right) \frac{dm}{dt} \qquad \text{(B.10)}$$

Introducing the notation r_c, r_e and v (absolute velocity) with the following equations (see Fig. 5)

$$x_1 - x_c = r_c - r_e, \quad v_1 = dr/dt - v$$

$$\frac{dx_1}{dt} = \frac{dr}{dt} - \frac{dr_e}{dt}, \quad \frac{dx_c}{dt} = \frac{dr}{dt} - \frac{dr_c}{dt} = u_c \qquad \text{(B.11)}$$

results in the equation of motion as usually specified.

$$0 = \frac{du_c}{dt} m + (r_c - r_e) \frac{d^2m}{dt^2} + \left(2 \frac{dr_c}{dt} - \frac{dr_e}{dt} - v\right) \frac{dm}{dt} \qquad \text{(B.12)}$$

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