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Space Guidance Analysis Memo #34-64

TO:	SGA Distribution
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DATE:	August 20, 1964
SUBJECT:	Orbital Navigation Using Unknown Landmarks.

1. Description of Navigation Procedure.

This memo presents a method of using sextant sightings to unknown landmarks for navigating in a near orbit of a planet. One measurement is considered to consist of two sightings to the same landmark. The position of the landmark (assumed constant for the moment) and the two points from which the sightings are made determine a plane. At one position between the two sighting points, the velocity vector of the spacecraft will be parallel to this plane.

It will be shown below that this position is exactly midway between the two sighting points if the orbit is circular - regardless of the location of the landmark. In addition, the deviation of this position from the midpoint for an elliptic orbit will be derived. This deviation will be negligible in most cases, but a first order approximation is given which will be sufficient for all possible situations.

Based on the above discussion, the following orbital navigation procedure is suggested:

At time t_0, let a sextant sighting be made to an unknown but recognizable landmark, and let $\underline{u}_0$ be the measured unit vector from the spacecraft to the landmark. Several seconds later, at time t_1, assume that a second sighting, $\underline{u}_1$, to the same landmark is obtained.

Let $\underline{r}_0$ and $\underline{v}_0$ be the position and velocity estimates stored in the computer on board the spacecraft at time t_0. The measurement data is incorporated as follows:

1. Integrate $\underline{r}_0$ and $\underline{v}_0$ ahead to the two times t_1 and

$$t_2 = \frac{1}{2}(t_0 + t_1) + \delta t \tag{1}$$

to obtain $\underline{r}_1$, $\underline{r}_2$ and $\underline{v}_2$. The deviation δt, if needed, is determined from $\underline{r}_0$, $\underline{v}_0$, and $\underline{r}_1$.

2. Use the position estimate $\underline{r}_1$ to modify $\underline{u}_1$ to $\underline{u}_1'$ in order to compensate for the rotation of the planet.

3. Calculate the unit normal to the plane of $\underline{u}_0$ and $\underline{u}_1'$ from

$$\underline{n} = \mathrm{UNIT}(\underline{u}_0 \times \underline{u}_1')$$

The six-dimensional geometry vector for the measurement is given by

$$\underline{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \underline{n} \end{pmatrix}$$

4. The component of the velocity vector in the direction of $\underline{n}$ is the measured quantity, the measured value is zero, and the estimated value is $\underline{n} \cdot \underline{v}_2$. Therefore, calculate the measured deviation from

$$\delta\tilde{v} = -\underline{n} \cdot \underline{v}_2$$

5. Update the estimates $\underline{r}_2$ and $\underline{v}_2$ and the W matrix at time t_2 in the usual manner.

Extrapolation now proceeds from time t_2.

2. Computer Simulation.

In order to determine the usefulness of the information obtained from the above orbital navigation procedure, the following two cases were considered:

1. An earth parking orbit prior to translunar injection of two and one half orbits or 3.75 hours in length. It was assumed that three landmarks could be observed in both Africa and North America during each of the two passes. Included in the simulation are the effects of venting, assumed to occur an average of nine times during the flight with a rectangularly distributed uncertainty having a maximum value of 0.7 ft/sec.

2. A lunar parking orbit prior to LEM descent of three orbits or 6.3 hours duration. Ten landmarks were observed at nine minute intervals during the first half of each orbit.

The table below shows the RMS terminal errors resulting from 25 Monte Carlo runs for each case. An RMS error of one minute was assumed in establishing each of the two sightings required for an observation.

In both cases, nearly perfect circular orbits were used as well as non-rotating planets. Elliptic orbits and rotating planets will not degrade the navigation provided proper compensation is included if required.

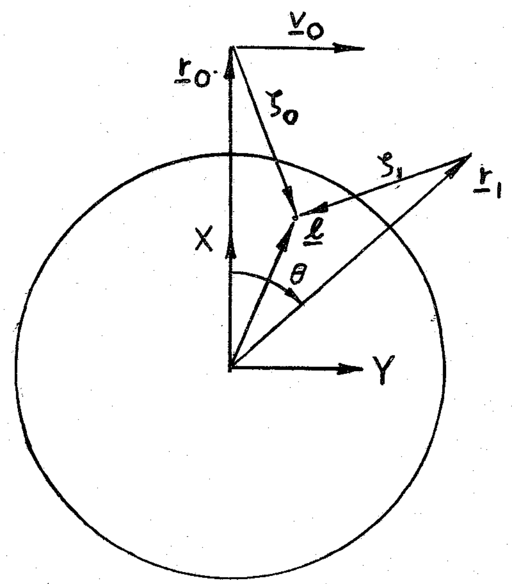
The landmarks were selected in a convenient, but not optimum, manner. Thus, it is probably possible to improve upon the results shown.

RMS Terminal Errors

	Position (mi)			Velocity (ft/sec)		
	Alt.	Range	Track	Alt.	Range	Track
Earth Orbit	0.9	2.3	0.9	13.4	5.6	6.3
Lunar Orbit	0.3	0.6	0.9	1.6	1.8	1.2

3. Circular Orbit.

Let $\underline{r}_0$ and $\underline{r}_1$ be the two points from which sightings to the landmark position $\underline{\ell}$ are obtained. Consider an inertial axis system which has its XY plane in the plane of $\underline{r}_0$ and $\underline{r}_1$ and its X axis along $\underline{r}_0$ as shown in the figure.



The position and velocity at time t_0 are given by

$$\underline{r}_0 = \begin{pmatrix} r_0 \\ 0 \\ 0 \end{pmatrix} \tag{2}$$

$$\underline{v}_0 = \begin{pmatrix} 0 \\ v_\theta \\ 0 \end{pmatrix} \tag{3}$$

If θ is the angle between $\underline{r}_0$ and $\underline{r}_1$, then the position at time t_1 is as follows

$$\underline{r}_1 = \begin{pmatrix} r_0 \cos \theta \\ r_0 \sin \theta \\ 0 \end{pmatrix} \tag{4}$$

Let the landmark be at the fixed arbitrary point

$$\underline{\ell} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{5}$$

The vectors from the two sighting points to the landmark are obtained from Eqs. (2), (4) and (5)

$$\underline{\rho}_0 = \underline{\ell} - \underline{r}_0 = \begin{pmatrix} x - r_0 \\ y \\ z \end{pmatrix} \tag{6}$$

$$\underline{\rho}_1 = \underline{\ell} - \underline{r}_1 = \begin{pmatrix} x - r_0 \cos\theta \\ y - r_0 \sin\theta \\ z \end{pmatrix} \tag{7}$$

A vector in the direction of the cross product of $\underline{\rho}_0$ and $\underline{\rho}_1$ is then given by

$$\underline{N} = \begin{pmatrix} \sin\theta \\ 1 - \cos\theta \\ \text{not important} \end{pmatrix} \tag{8}$$

It is desired to find the point $\underline{r}$ at angle ϕ to $\underline{r}_0$ such that the velocity $\underline{v}$ is normal to $\underline{N}$. The velocity $\underline{v}$ has the following components:

$$\underline{v} = \begin{pmatrix} -v_\theta \sin\phi \\ v_\theta \cos\phi \\ 0 \end{pmatrix} \tag{9}$$

Taking the dot product of Eq. (9) with Eq. (8) yields

$$\underline{v} \bullet \underline{N} = v_\theta(-\sin\theta \sin\phi + \cos\phi - \cos\theta \cos\phi)$$

$$= v_\theta \left[\cos\phi - \cos(\theta - \phi)\right]$$

which equals zero if

$$\phi = \frac{\theta}{2}$$

as was to be shown.

4. Elliptic Orbit.

Let the velocity at time t_0 be given by

$$\underline{v}_0 = \begin{pmatrix} v_R \\ v_\theta \\ 0 \end{pmatrix} \tag{10}$$

The position at time t_1 is as follows:

$$\underline{r}_1 = \begin{pmatrix} r_1 \cos\theta \\ r_1 \sin\theta \\ 0 \end{pmatrix} \tag{11}$$

Differencing Eqs. (5) and (11) yields

$$\underline{\rho}_1 = \underline{\ell} - \underline{r}_1 = \begin{pmatrix} x - r_1 \cos\theta \\ y - r_1 \sin\theta \\ z \end{pmatrix} \tag{12}$$

A vector normal to the plane of $\underline{\rho}_0$ and $\underline{\rho}_1$ is then obtained from Eqs. (6) and (12).

$$\underline{N} = \begin{pmatrix} \sin\theta \\ \frac{r_0}{r_1} - \cos\theta \\ — \end{pmatrix} \tag{13}$$

To determine the ratio r_0/r_1 use Eqs. (B1.46),* (2), and (10) to write

$$r_1 = \frac{p}{1 + (p/r_0 - 1)\cos\theta - \sqrt{p/\mu}\, v_R \sin\theta} \tag{14}$$

Now, from Eq. (B1.34) and Eqs. (2) and (10)

$$p = \frac{h^2}{\mu} = \frac{r_0^2 v_\theta^2}{\mu} = r_0 \frac{v_\theta^2}{v_c^2} \tag{15}$$

where v_c is the circular orbital speed. Using Eq. (15) in Eq. (14) yields

$$\frac{r_0}{r_1} = \cos\theta - \frac{v_R}{v_\theta}\sin\theta + \frac{v_c^2}{v_\theta^2}(1 - \cos\theta) \tag{16}$$

Then, substituting Eq. (16) into Eq. (13) gives

$$\underline{N} = \begin{pmatrix} \sin\theta \\ -\frac{v_R}{v_\theta}\sin\theta + \frac{v_c^2}{v_\theta^2}(1 - \cos\theta) \\ — \end{pmatrix} \tag{17}$$

The velocity vector $\underline{v}$ at point $\underline{r}$ is obtained from Eq. (B1.45) and Eqs. (2) and (10). Thus

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* Equations referred to in this manner are from Battin, R. H., Astronautical Guidance.

$$\underline{v} = \left[\frac{v_R}{p}(1 - \cos\phi) - \frac{1}{r_0}\sqrt{\frac{\mu}{p}}\sin\phi\right]\underline{r}_0$$

$$+ \left[1 - \frac{r_0}{p}(1 - \cos\phi)\right]\underline{v}_0 \qquad (18)$$

Using Eqs. (2), (10), and (15) in Eq. (18) gives

$$\underline{v} = v_\theta \begin{pmatrix} \dfrac{v_R}{v_\theta} - \dfrac{v_c^2}{v_\theta^2}\sin\phi \\ 1 - \dfrac{v_c^2}{v_\theta^2}(1 - \cos\phi) \\ 0 \end{pmatrix} \qquad (19)$$

Taking the dot product of Eqs. (17) and (19) yields

$$\underline{v} \cdot \underline{N} = \frac{v_c^2}{v_\theta}\left[\cos\phi - \cos(\theta - \phi) + \alpha(1 - \cos\phi)\right] \qquad (20)$$

where

$$\alpha = \frac{v_R}{v_\theta}\sin\theta + \left(1 - \frac{v_c^2}{v_\theta^2}\right)(1 - \cos\theta) \qquad (21)$$

Let the value of ϕ, which makes Eq. (20) vanish, be

$$\phi = \frac{\theta}{2} + \delta\phi \qquad (22)$$

Substituting Eq. (22) into Eq. (20) and setting the result equal to zero gives

$$\sin \delta\phi = \frac{\alpha}{2 - \alpha} \; \frac{1 - \cos \dfrac{\theta}{2} \cos \delta\phi}{\sin \dfrac{\theta}{2}} \tag{23}$$

Equation (22) is an exact expression for $\delta\phi$. Since the angle θ itself is not large, the deviation $\delta\phi$ is sufficiently small so that $\sin \delta\phi$ and $\cos \delta\phi$ may be replaced by $\delta\phi$ and unity in Eq. (22). This approximation yields

$$\delta\phi = \frac{\alpha}{2 - \alpha} \tan \frac{\theta}{4} \tag{24}$$

Equation (21) is a convenient expression for calculating α in an on-board computer since $\underline{r}_0$ and $\underline{v}_0$ are immediately available. For the purposes of analysis, however, the following more convenient form of Eq. (21) is obtained using Eqs. (B1.33), (B1.34), and (B1.39).

$$\alpha = e \frac{\cos f_0 - \cos (f_0 + \theta)}{1 + e \cos f_0} \tag{25}$$

To determine the necessity of implementing the deviation δt, write Eq. (25) to first order in θ as follows

$$\alpha = e\theta \frac{\sin f_0}{1 + e \cos f_0} \tag{26}$$

Equation (26) has a maximum value given by

$$\alpha_{MAX} = \frac{e\theta}{\sqrt{1 - e^2}} \tag{27}$$

Substituting Eq. (27) into Eq. (24) yields

$$\delta\phi_{MAX} = \frac{e\theta^2}{4(2\sqrt{1 - e^2} - e\theta)}$$

as the maximum value of $\delta\phi$ to first order in θ. For all possible cases, the denominator of Eq. (28) has a value greater than 7.9. Thus,

$$\delta\phi_{MAX} = \frac{e\theta^2}{7.9}$$

Assuming a maximum angle between the measurement plane and the vertical of 45°, the maximum velocity error which results if $\delta\phi$ is ignored is given by

$$\delta v_{MAX} = \frac{e\theta^2}{11} v$$

where v is the magnitude of the velocity vector. Using 0.05 and 0.2 as the maximum values of θ for earth orbit and lunar orbit respectively yields a velocity error bound in feet per second given by

$$\delta v_{MAX} = \begin{cases} 6e & \text{for earth orbit} \\ 20e & \text{for lunar orbit} \end{cases}$$

Although the errors displayed in Eq. (31) may not be negligible for all possible orbits, the approximation

$$\delta\phi = \frac{1}{8}\theta^2 \frac{v_R}{v_\theta}$$

can be shown to yield negligible errors. Hence, the time variation δt in Eq. (1) can be calculated from

$$\delta t = \frac{1}{8}\theta \frac{v_R}{v_\theta}(t_1 - t_0)$$

or

$$\delta t = \frac{1}{8}\arccos\left(\frac{\underline{r}_0 \cdot \underline{r}_1}{r_0 r_1}\right)\frac{\underline{r}_0 \cdot \underline{v}_0}{|\underline{r}_0 \times \underline{v}_0|}(t_1 - t_0)$$