CUBIC SPLINE FUNCTION INTERPOLATION IN ATMOSPHERE MODELS FOR THE SOFTWARE DEVELOPMENT LABORATORY: FORMULATION AND DATA

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16. Abstract

A tabulation of selected altitude-c lated values of pressure, density, speed of sound, and coefnicient of viscosity for each of ins models of the atmosphere is presented in block data format. Interpolation for the desired atmospheric parameters is performed by using cubic spline functions. The recursive relations necessary to compute the cubic spline function coefficients are derived and implemented in subrcutine form. Tiree companion subprograms, which form the preprocessor and processor, are also presented. These subprograms, together with the data element, compose the spline fit atmosphere package. Detailed FLOWGM flow charts and FORTRAN listings of the atmosphere package are presented in the appendix.


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# CUBIC SPLINE FUNCTION INTERPOLATION IN ATMOSPHERE MODELS FOR THE SOFTWARE DEVELOPMENT LABORATORY: <br> FORMULATION AND DATA <br> By James C. Kirkpatrick <br> Lyndon B. Johnson Space Center 

## SUMMARY

Six standard reference atmosphere models for the years 1962, 1963 (Patrick Air Force Base), and 1966 (for July and January at latitudes $30^{\circ}$ and $60^{\circ} \mathrm{N}$ ) are presented in tabular form. The tabulation, which is presented in block data format, is adequate for the accurate representation of the atmospheric parameters of pressure, density, speed of sound, and coefficient of viscosity as functions of altitude. The range of tabulated altitudes extends from 0 to 205 kilometers. Interpolation for the desired parameters is performed by using cubic spline functions. The recursive relations necessary to compute the cubic spline function coefficients are derived and implemented in subroutine form. Detailed FLOWGM flow charts and FORTRAN listings of the atmosphere package are presented in the appendix.

## INTRODUCTION

This report presents a tabulation in block data format of selected altitudecorrelated values of pressure, density, speed of sound, and coefficient of viscosity for each of six atmosphere models. These values were selected such that they adequately represent their respective functions for accurate interpolation by cubic spline functions throughout the range of tabulated altitudes. Three companion subprograms, which form the preprocessor (subroutines SDAT and SPLN1) and processor (subroutine ATMSFL), are presented. These subprograms, together with the data element, compose the spline fit atmosphere package. In the preprocessor, the data for the desired atmosphere model are selected and processed for proper order and unit consistency before SPLN 1 is used to compute the spline function coefficients. In the processor, the data table is searched to establish the interval in which the desired altitude lies, and then the values of the previously mentioned parameters are computed by using the cubic spline function coefficients for the appropriate altitude interval.

The cubic spline technique provides an effective, easy-to-use method for the unerring reproduction of a function from tabulated data. The method also ensures continuity in the function and in its first and second derivatives. After
the cubic spline function coefficients hav : been computed, they may be stored for repeated use as long as the tabulated data remain unchanged.

## SYMBULS

| a, b, c, d | edges of the beam |
| :---: | :---: |
| $a_{k}$ | defined in equation (37) |
| $\mathrm{b}_{\mathbf{k}}$ | defined in equation ! 38) |
| $C_{1}, C_{2}$ | arbitrary constants of integration |
| $c_{k}$ | defined in equation (39) |
| $c_{1}(k)$ | defined in equation (68) |
| $\mathrm{C}_{2}(\mathrm{k})$ | defined in equation (69) |
| $c_{3}(\mathrm{k})$ | defined in equation (70) |
| dA | differential element of a cross-sectional area of a beam |
| ds | differential element of arc length |
| dx | differential element of the length of the beam |
| d $\theta$ | differential element of the angle through which a cross-sectional surface of a beam is rotated under load |
| $\mathrm{d}_{k}$ | defined in equation (49) |
| E | modulus of elasticity of the beam |
| EI | flexural rigidity (can be taken as unity for any plane curve without loss of generality) |
| ef | original unstressed length of the fiber |
| gh | typical fiber of the beam |
| hk | defined in equation (1) |
| I | area moment of inertia |
| $\ell$ | distance from the neutral surface to a deformed fiber of a loaded beam |


| M ( x ) | bending moment as a function of the distance $x$ measured from the end of the beam |
| :---: | :---: |
| $M_{k}$ | bending moment at the kth section of the beam or an approximation to the second derivative at the kth section |
| P | load that acts upon the beam |
| $Q_{k}$ | defined in equation (44) |
| $\mathrm{R}_{1}, \mathrm{R}_{2}$ | support reactions of the beam |
| S | stress |
| s | arc length of a curve |
| $\mathrm{U}_{\mathrm{k}}$ | defined in equation (46) |
| $\mathrm{V}_{\mathrm{k}}$ | shear force at the kth section of the beam |
| $y(k)$ | defined in equation (24) |
| $y(x)$ | defined in equation (23) |
| $y^{\prime}$ | slope of the deformation curve |
| $y^{\prime}(x)$ | defined in equation (22) |
| $\varepsilon$ | strain (change in length per unit of length) |
| $\rho$ | radius of curvature of the neutral surface |
| $\varphi$ | angle that the tangent to a curve makes with the axis of abscissas |

## SPLINE FIT ATMOSPHERE MODEL PACKAGE

## Model History

The spline fit atmosphere package discussed in this report, was originally developed for the space vehicle dynamic simulator (SVDS) program in November 1972. In February 1973, the package was incorporated with appreciable modification, into the Shuttle optimal abort program (SOAP). The package presented here is from an early version of the SOAP. For implementation into the SOAP, the SVDS version was modified, at the expense of storage, to improve its performance and execution requirements. The spline fit atmosphere package has performed accurately in both programs.

The spline fit atmosphere package consists of six atmosphere models:

1. The 1962 standard reference atmosphere
2. The 1966 standard reference atmosphere for July at latitude $30^{\circ} \mathrm{N}$
3. The 1966 standard reference atmosphere for January at latitude $30^{\circ} \mathrm{N}$
4. The 1966 standard reference atmosphere for July at latitude $60^{\circ} \mathrm{N}$
5. The 1966 standard reference atmosphere for January at latitude $60^{\circ} \mathrm{N}$
6. The 1963 Patrick Air Force Base (AFB) reference atmosphere

Each model consists of a tabulation of altitude-correlated values of the atmospheric parameters of pressure, density, speed of sound, and coefficient of viscosity. All six models are correlated to the same table of altitudes and tabulated in the foregoing sequence. All tabulated atmospheric parameter values are expressed in the meter-kilogram-second (MKS) system of units. Altitude values are given in meters, pressure values are given in millibars, density values are given in kilograms per cubic meter, speed of sound values are given in meters per second, and coefficient of viscosity values are given in kilograms per meter per second. Each atmospheric parameter is represented as a function of altitude by 123 data points. These data have been selected on best-fit considerations, with particular emphasis on those regions in which the tabulation shows that the function is discontinuous in its first derivative. Each model is represented in the 0- to 205 -kilometer altitude range. The tabulation was compiled from references 1 to 3 .

Because the tabulation for the 1966 atmosphere extends only $t / 118$ kilometers, the last data point is repeated to 205 kilometers for model consistency. If the model altitude bounduries are exceeded, the package returns the last tabulated parameter values at the violated boundary.

## Assumptions and Approximations

The spline fit atmosphere model package was developed to accommodate the needs and restrictions of the SVDS program. As a result, temperature, an important atmospheric parameter, was not included in the list of parameters modeled when the package was built. This parameter can be added to any model desired at the cost of tabulating the parameter values required.

The tabulation was performed in the MKS system of units because this system was common to al the references used. A common system of units is advantageous because a single set of conversion factors can be used to place all models in any desired working units. The British engineering system of units was used to convert the atmospheric data in this package (i.e., pressure in pounds-force per square foot, density in slugs per cubic foot, speed of sound in feet per second, and coefficient of viscosity in pounds-mass seconds per square foot).

Atmospheric parameter function values were selected at various altitudes. The increment of altitude was varied as a function of altitude as follows.

| Altitude range, km | Altitude increment, m |
| :---: | :---: |
| 0 to 3 | 250 |
| 3 to 5 | 500 |
| 5 to 20 | 1000 |
| 20 to 160 | 2000 |
| 160 to 204 | 4000 |
| 204 to 205 | 1000 |

Additional data points were added to mark the location in which the atmospheric parameter functions undergo first-derivative discontinuities. These discontinuities occur only in the speed of sound and coefficient of viscosity functions, which are affected because of temperature variations. As a result, when temperature is added to the atmosphere models, the choice of altitudes should accommodate discontinuities in the first derivative of the temperature functions. Because of the need to intersperse additional points, an ordering scheme is included in the preprocessor SDAT to place all altitude values and associated function values in ascending order of the independent variable. This order must be established before calling subroutine SPLN1 to compute the cubic spline coefficients.

The small altitude increment used at the extremes of the tabulation permitted the approximation of the first derivatives of the function at the initial and terminal boundaries by forming the difference quotient. This assumption of linearity at the extremes of the tabulated function is the only assumption made in the model. (The theoretical arguments on which the algorithm is based result in the additional assumption that the curvature of the function can be represented by its second derivative. The data sample selected is expected to comply with this assumption.)

## DERIVATION OF THE RECURSIVE RELATIONS FOR COMPUTING THE COEFFICIENTS OF THE CUBIC SPLINE FUNCTIONS

The mathematical spline is a formal analog of the draftsman's spline, a flexible beam device used to fair out a smooth curve between specified data points. The mechanical spline is anchored on the drawing board at each plotted data point. The curvature of the spline results from the bending moments applied to the beam.

In deriving the recursive relations of the spline algorithm, the elementary theory of elasticity is useful. Consider a simply supported beam (i.e., ends
unrestrained) under the action of an applied load, however distributed. For an elementary analysis, the following assumptions are made:

1. Sections of the beam, which were originally plane, remain plane.
2. The material of the beam is homogeneous and obeys Hooke's law (the deformation is proportional to the applied load).
3. The moduli of elasticy for tension and for compression are equal.
4. The beam is initially straight and is of constant cross section.

Figure 1 shows a uniform homogeneous beam under the action of the load $P$. The beam is simply supported and is in static equilibrium under the action of the support reactions $R_{1}$ and $R_{2}$. The differential element of length $d x$, bounded by
the eages $a, b, c$, and $d$, is bent or deformed under load into the exaggerated shape shown in figure 2 where $c$ and $d$ are deflected to $c^{\prime}$ and $d^{\prime}$. The sections $a b$ and cd are shown rotated relative to each other through an angle $d \theta$, but they remain undistorted in accordance with assumption 1. The fibers in the region of ac are contracted, or shortened, in compression, whereas those in the region bd are elongated, or lengthened, in tension. Because the beam is uniform and homogeneous in going from contraction to elongation, a region conceivably exists in which the fibers maintain their original unstressed length. This surface of unstressed fibers is referred to as the neutral surface. In the deformation of a typical fiber gh located at distance $\ell$ from the neutral surface, the elongation hk is the arc of a circle of radius $\ell$, which subtends the angle $d \theta$ and is given by the relation

$$
\begin{equation*}
\mathbf{h k}=\ell \mathbf{d} \theta \tag{1}
\end{equation*}
$$

The strain $\varepsilon$ (the change in leagth per unit of length) is found by dividing the deformation hk by the original unstressed length of the fiber ef.

$$
\begin{equation*}
\varepsilon=\frac{\mathrm{hk}}{\mathrm{ef}}=\frac{\ell \mathrm{d} \theta}{\mathrm{ef}} \tag{2}
\end{equation*}
$$

If the radius of curvature of the netitral surfe $2 e$ is $\rho$, the curved length of ef is equal to $\rho \mathrm{d} \theta$. The strain can ther be expressed as

$$
\begin{equation*}
\varepsilon=\frac{\ell d \theta}{\rho d \theta}=\frac{\ell}{\rho} \tag{3}
\end{equation*}
$$

Because the material is assumed to be homogeneous and to obey Hooke's law, the stress (force per unit area) in the fiber gh is given by

$$
\begin{equation*}
S=E_{\varepsilon}=\frac{E}{\rho} \ell \tag{4}
\end{equation*}
$$

where $E$ is the modulus of elasticity defined as the ratio of the stress $S$ to the $\operatorname{strain} \varepsilon$. Clearly, the sign of $S$ varies with the sign of $\ell$ : negative or compressive stress for $\ell$ negative and positive or tensile stress for $\ell$ positive.

The beam in figure 1 is shown in static equilibrium. As a result, the algebraic sum of all forces and moments acting on the beam must be zero. The bending moment $M(x)$ must be balanced by the resisting moment. The resisting moment is the sum of all stresses S acting on each differential area element of cross section $d A$ multiplied by its moment arm $\ell$ measured from the neutral axis as shown in figure 3. Hence, the bending moment is

$$
\begin{equation*}
M(x)=\int \ell(S d A) \tag{5}
\end{equation*}
$$

Replacing $S$ with $E \ell / \rho$ from equation (4) gives

$$
\begin{equation*}
M(x)=E I / \rho \tag{6}
\end{equation*}
$$

where I is the area moment of inertia.

$$
\begin{equation*}
\frac{1}{\rho}=\frac{M(x)}{E I} \tag{7}
\end{equation*}
$$

The reciprocal of $\rho$, the radius of curvature of the beam, is called the curvature of the curve. For plane curves, the curvature is defined as the rate at which the tangent to the curve turns compared with the description of arc. Therefore. if $\varphi$ is the angle that the tangent to a curve makes with the axis of abscissas and $s$ is the element of arc length, then the curvature is defined as

$$
\begin{equation*}
\frac{1}{\rho}=\frac{d \varphi}{d s} \tag{8}
\end{equation*}
$$

The slope of the tangent to the curve is

$$
y^{\prime}=\frac{d y}{d x}=\tan \varphi
$$

or

$$
\begin{equation*}
\varphi=\arctan y^{\prime} \tag{10}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{d} \varphi=\frac{\mathrm{d} \mathrm{y}^{\prime}}{\left(1+\mathrm{y}^{\prime-)^{\frac{1}{2}}}\right.} \mathrm{dx}=\frac{\mathrm{y}^{\prime \prime}}{\left(1+\mathrm{y}^{\prime}\right)^{\frac{1}{2}}} \mathrm{dx} \tag{11}
\end{equation*}
$$

The differential element of arc length ds is defined as

$$
\begin{equation*}
d s=\left(1+y^{\prime}\right)^{\frac{1}{2}} d x \tag{12}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\frac{1}{\rho}=\frac{d \varphi}{d s}=\frac{y^{\prime \prime}}{\left(1+y^{\prime}\right)^{\frac{3}{2}}} \tag{13}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{y^{\prime \prime}}{\left(1+y^{\prime}\right)^{\frac{3}{2}}}=\frac{M(x)}{E I} \tag{14}
\end{equation*}
$$

The assumption is usually made that the slope of the deformation curve $y^{\prime}$ is small so that $\mathrm{y}^{\prime 2}$ can be ignored compared to unity. When this assumption is made, the differential equation of the deformation curve can be written as

$$
\begin{equation*}
M(x)=\text { Ely' }(x) \tag{15}
\end{equation*}
$$

The assumption concerning the elastic behavior of a beam made at the beginning of this discussion can be realized by a mathematical function of one variable. The assumption that the slope of the curve is small and that its square can be neglected can be realized by an arbitrary mathematical function only over relatively short intervals. Under this condition, the curvature can be approximated by the second derivative of the function. This point must be remembered when selecting a set of data points to represent a function for spline fit. Sufficient data points must be chosen in the appropriate concentration such that in the regions where the curvature is large, each section of the curve can be considered as a simply supported seam (adequately supported so that the slope of the deformation curve $s$ negligible).

The fcllowing definitions are required for the discussion that follows.

1. The shear at a section of a beam is the aigebraic sum of all external forces acting on one side of the section.
2. The bending moment at a section of a beam is the algebraic sum of the moments of all the external forces on one side of the section.

Hence, if a mathematical function is represented in a region of definition by $n$ data points, the region of definition will have been divided into $n-1$ sections (fig. 4). If each section of the curve between given data points is considered to be a beam deflected under load, then the bending moment in the kth section between the kth and $(k+1)$ th data points will be given by the equatic

$$
\begin{equation*}
M(x)=\int_{x_{k}}^{x} V_{k} d x+M_{k} \tag{i1.}
\end{equation*}
$$

where $V_{k}$ and $M_{k}$ are the shear and bending moments at the kth section. Therefore,

$$
\begin{equation*}
M(x)=V_{k}\left(x-x_{k}\right)+M_{k} \tag{17}
\end{equation*}
$$

Clearly, where $x=x_{k}, M\left(x_{k}\right)=M_{k}$ and where $x=x_{k+1}, M\left(x_{k+1}\right)=M_{k+1}$ or

$$
\begin{equation*}
M_{k+1}=V_{k}\left(x_{k+1}-x_{k}\right)+M_{k} \tag{18}
\end{equation*}
$$

Solving for $\mathrm{V}_{\mathrm{k}}$ gives

$$
\begin{equation*}
\mathbf{V}_{\mathbf{k}}=\frac{\mathrm{M}_{\mathbf{k}+1}-\mathrm{M}_{\mathbf{k}}}{\mathbf{x}_{\mathbf{k}+1}-\mathbf{x}_{\mathbf{k}}}=\frac{\Delta \mathbf{M}_{\mathbf{k}}}{\Delta \mathbf{x}_{\mathbf{k}}} \tag{19}
\end{equation*}
$$

Substituting equation (19) into equation (17) gives

$$
\begin{align*}
M(x) & =\frac{\left(M_{k+1}-M_{k}\right)}{\left(x_{k+1}-x_{k}\right)}\left(x-x_{k}\right)+M_{k} \\
& =M_{k+1} \frac{\left(x-x_{k}\right)}{\left(x_{k+1}-x_{k}\right)}+R \cdot \frac{\left(x_{k+1}-x\right)}{\left(x_{k+1}-x_{k}\right)} \tag{20}
\end{align*}
$$

Substituting the bending moment $M(x)$ by its equivalent from equation (15) (ihe prrduct El is known as the flexural rigidity and can be taken as unity for any plane curve without loss of generality) in equation (20) gives

$$
\begin{equation*}
y^{\prime \prime}(x)=M_{k+1} \frac{\left(x-x_{k}\right)}{\Delta x_{k}}+M_{k} \frac{\left(x_{k+1}-x\right)}{\Delta x_{k}} \tag{21}
\end{equation*}
$$

where $\Delta x_{k}=x_{k+1}-x_{k}$. Integrating twice gives

$$
\begin{align*}
& y^{\prime}(x)=M_{k+1} \frac{\left(x-x_{k}\right)^{2}}{2 \Delta i_{k}}-M_{k} \frac{\left(x_{k+1}-x\right)^{2}}{2 \Delta x_{k}}+C_{1}  \tag{22}\\
& y(x)=M_{k+1} \frac{\left(x-x_{k}\right)^{3}}{6 \Delta x_{k}}+M_{k} \frac{\left(x_{k+1}-x\right)^{3}}{6 \Delta x_{k}}+C_{1} x+C_{2} \tag{23}
\end{align*}
$$

where $C_{1}$ and $C_{2}$ are constants of integration. At $x=x_{k}, y\left(x_{k}\right)=y_{k}$; at $x=x_{k+1}, y\left(x_{k+1}\right)=y_{k+1}$. Substituting $y_{k}$ and $y_{k+1}$ in equation (23) gives

$$
\begin{equation*}
y_{k}=M_{k} \frac{\Delta x_{k}^{2}}{6}+C_{1} x_{k}+C_{2} \tag{24}
\end{equation*}
$$

and

$$
\begin{equation*}
y_{k+1}=M_{k+1} \frac{\Delta x_{k}^{2}}{6}+C_{1} x_{k+1}+C_{2} \tag{25}
\end{equation*}
$$

Subtracting equation (25) from equation (25) gives

$$
\begin{equation*}
y_{k+1}-y_{k}=\left(M_{k+1}-M_{k}\right) \frac{\Delta x_{k}^{2}}{6}+C_{1} \Delta x_{k} \tag{26}
\end{equation*}
$$

Solving for $C_{1}$ from equation (26) gives

$$
\begin{align*}
C_{1} & =\frac{y_{k+1}-y_{k}}{\Delta x_{k}}-\frac{\left(M_{k+1}-M_{k}\right) \Delta x_{k}}{6} \\
& =\left(\frac{y_{k+1}}{\Delta x_{k}}-\frac{M_{k+1} \Delta x_{k}}{6}\right)-\left(\frac{y_{k}}{\Delta x_{k}}-\frac{M_{k} \Delta x_{k}}{6}\right) \tag{27}
\end{align*}
$$

Substituting for $C_{1}$ from equation (27) in equation (25) and solving for $C_{2}$ gives

$$
\begin{equation*}
c_{2}=x_{k+1}\left(\frac{y_{k}}{\Delta x_{k}}-\frac{M_{k} \Delta x_{k}}{6}\right)-x_{k}\left(\frac{y_{k+1}}{\Delta x_{k}}-\frac{M_{k+1} \Delta x_{k}}{6}\right) \tag{28}
\end{equation*}
$$

Substituting the expressions for $C_{1}$ and $C_{2}$ from equations (27) and (28) in equations (22) and (23) gives the expression for the cubic spline function in the kth interval as

$$
\begin{align*}
y(x)= & M_{k+1} \frac{\left(x-x_{k}\right)^{3}}{6 \Delta x_{k}}+M_{k} \frac{\left(x_{k+1}-x\right)^{3}}{6 \Delta x_{k}} \\
& +\left(\frac{y_{k+1}}{\Delta x_{k}}-\frac{M_{k+1} \Delta x_{k}}{6}\right)\left(x-x_{k}\right)+\left(\frac{y_{k}}{\Delta x_{k}}-\frac{M_{k} \Delta x_{k}}{6}\right)\left(x_{k+1}-x\right) \tag{29}
\end{align*}
$$

and its first derivative is given by

$$
\begin{align*}
y^{\prime}(x)= & M_{k+1} \frac{\left(x-x_{k}\right)^{2}}{2 \Delta x_{k}}-M_{k} \frac{\left(x_{k+1}-x\right)^{2}}{2 \Delta x_{k}} \\
& +\left(\frac{y_{k+1}}{\Delta x_{k}}-\frac{M_{k+1} \Delta x_{k}}{6}\right)-\left(\frac{y_{k}}{\Delta x_{k}}-\frac{M_{k} \Delta x_{k}}{6}\right) \tag{30}
\end{align*}
$$

Equations (21), (29), and (30) provide expressions for the values of the cubic spline function and its first and second derivatives in the kth interval. These expressions can be written for the cubic spline function and its first and second derivatives in the $(k-1)$ th interval by replacing $k$ with $k-1$. Therefore, in the ( $k-1$ )th interval,

$$
\begin{align*}
y(x)= & M_{k} \frac{\left(x-x_{k-1}\right)^{3}}{6 \Delta x_{k-1}}+M_{k-1} \frac{\left(x_{k}-x\right)^{3}}{6 \Delta x_{k-1}} \\
& +\left(\frac{y_{k}}{\Delta x_{k-1}}-\frac{M_{k} \Delta x_{k-1}}{6}\right)\left(x-x_{k-1}\right) \\
& +\left(\frac{y_{k-1}}{\Delta x_{k-1}}-\frac{M_{k-1} \Delta x_{k-2}}{6}\right)\left(x_{k}-x\right)  \tag{31}\\
y^{\prime}(x)= & M_{k} \frac{\left(x-x_{k-1}\right)^{2}}{2 \Delta x_{k-1}}-M_{k-1} \frac{\left(x_{k}-x\right)^{2}}{2 \Delta x_{k-1}} \\
& +\left(\frac{y_{k}}{\Delta x_{k-1}}-\frac{M_{k} \Delta x_{k-1}}{6}\right)-\left(\frac{y_{k-1}}{\Delta x_{k-1}}-\frac{M_{k-1} \Delta x_{k-1}}{6}\right)  \tag{32}\\
y^{\prime \prime}(x)= & M_{k} \frac{\left(x-x_{k-1}\right)}{\Delta x_{k-1}}+M_{k-1} \frac{\left(x_{k}-x\right)}{\Delta x_{k-1}} \tag{33}
\end{align*}
$$

A cursory inspection of equations (29) and (31) and equations (21) and (33) will show that neither the cubic spline function nor its second derivative will experience discontinuity in going from the ( $k-1$ ) th to the $k$ th interval. This may be shown simply be replacing $x$ with $x_{k}$ in these expressions. To ensure that the cubic spline functions will experience no discontinuity in the first derivative in going from the $(k-1)$ th to the $k$ th interval, replace $x$ with $x_{k}$ in equations (30) and (32) and equate the results. Thus,

$$
\begin{array}{r}
M_{k} \frac{\Delta x_{k-1}}{2}+\left(\frac{y_{k}}{\Delta x_{k-1}}-\frac{M_{k} \Delta x_{k-1}}{6}\right)-\left(\frac{y_{k-1}}{\Delta x_{k-1}}-\frac{M_{k-1} \Delta x_{k-1}}{6}\right) \\
\quad=-M_{k} \frac{\Delta x_{k}}{2}+\left(\frac{y_{k+1}}{\Delta x_{k}}-\frac{M_{k+1} \Delta x_{k}}{6}\right)-\left(\frac{y_{k}}{\Delta x_{k}}-\frac{M_{k} \Delta x_{k}}{6}\right) \tag{34}
\end{array}
$$

or

$$
\begin{equation*}
\frac{\Delta x_{k-1}}{6} M_{k-1}+\frac{\Delta x_{k-1}+\Delta x_{k}}{3} M_{k}+\frac{\Delta x_{k}}{6} M_{k+1}=\frac{\Delta y_{k}}{\Delta x_{k}}-\frac{\Delta y_{k-1}}{\Delta x_{k-1}} \tag{35}
\end{equation*}
$$

where $\Delta y_{k}=y_{k+1}-y_{k}$ and $\Delta y_{k-1}=y_{k}-y_{k-1}$. Equation (35) permits the writing of $k$ equations for the solution of $k+2$ unknown values of $M$. Clearly, two second derivative values must be supplied at any two of the tabulated data points. However, if the second derivatives are not known, first derivative values may be used through equation (30). Further, if both the first and second derivatives of the function are known at any one of the tabulated points, the set of simultaneous equations given by equation (35) can still be solved.

Solving for $M_{k+1}$ in equation (35) gives

$$
\begin{align*}
M_{k+1} & =-\frac{2\left(\Delta x_{k}+\Delta x_{k-1}\right) M_{k}}{\Delta x_{k}}-\frac{\Delta x_{k-1}}{\Delta x_{k}} M_{k-1}+\frac{6}{\Delta x_{k}}\left(\frac{\Delta y_{k}}{\Delta x_{k}}-\frac{\Delta y_{k-1}}{\Delta x_{k-1}}\right) \\
& =a_{k} M_{k}+b_{k} M_{k-1}+c_{k} \tag{36}
\end{align*}
$$

where

$$
\begin{align*}
& a_{k}=-\frac{2\left(\Delta x_{k}+\Delta x_{k-1}\right)}{\Delta x_{k}}  \tag{37}\\
& b_{k}-\frac{\Delta x_{k-1}}{\Delta x_{k}}  \tag{38}\\
& c_{k}=\frac{6}{\Delta x_{k}}\left(\frac{\Delta y_{k}}{\Delta \lambda_{k}}-\frac{\Delta y_{k-1}}{\Delta x_{k-1}}\right) \tag{39}
\end{align*}
$$

Assume a solution of the form

$$
\begin{equation*}
M_{k+1}=\frac{1}{Q_{k}}\left(M_{k}-U_{k}\right) \tag{40}
\end{equation*}
$$

so that in the $(k-1)$ th interval

$$
\begin{equation*}
M_{k-1}=Q_{k-1} M_{k}+U_{k-1} \tag{41}
\end{equation*}
$$

Substituting equation (41) in equation (36) gives

$$
\begin{align*}
M_{k+1} & =a_{k} M_{k}+b_{k}\left(Q_{k-1} M_{k}+U_{k-1}\right)+c_{k} \\
& =\left(a_{k}+b_{k} Q_{k-1}\right) M_{k}+b_{k} U_{k-1}+c_{k} \tag{42}
\end{align*}
$$

Comparing equations (40) and (42) gives

$$
\begin{equation*}
\frac{1}{Q_{k}}=a_{k}+b_{k} Q_{k-1} \tag{43}
\end{equation*}
$$

or

$$
\begin{equation*}
Q_{k}=\frac{1}{a_{k}+b_{k} Q_{k-1}} \tag{44}
\end{equation*}
$$

and

$$
\begin{equation*}
-\frac{U_{k}}{Q_{k}}=b_{k} U_{k-1}+c_{k} \tag{45}
\end{equation*}
$$

or

$$
\begin{equation*}
U_{k}=-Q_{k}\left(b_{k} U_{k-1}+c_{k}\right) \tag{46}
\end{equation*}
$$

Substituting equations (37), (38), and (39) in equations (44) and (46) gives

$$
\begin{gather*}
Q_{k}=-\frac{\Delta x_{k}}{2\left(\Delta x_{k}+\Delta x_{k-1}\right)+\Delta x_{k-1} Q_{k-1}}=-\frac{\Delta x_{k}}{2\left(x_{k+1}-x_{k-1}\right)+\Delta x_{k-1} Q_{k-1}}  \tag{47}\\
U_{k}=-Q_{k}\left[-\frac{\Delta x_{k-1}}{\Delta x_{k}} U_{k-1}+\frac{6}{\Delta x_{k}}\left(\frac{\Delta y_{k}}{\Delta x_{k}}-\frac{\Delta y_{k-1}}{\Delta x_{k-1}}\right)\right] \\
=\frac{Q_{k}}{\Delta x_{k}}\left[\Delta x_{k-1} U_{k-1}+6\left(\frac{\Delta y_{k-1}}{\Delta x_{k-1}}-\frac{\Delta y_{k}}{\Delta x_{k}}\right)\right] \tag{48}
\end{gather*}
$$

Equations (40), (41), (47), and (48) are the recursive relations that will be used for the solution of the set of simultaneous linear equations given by equation (35). Before this is done, equations (29) and (30) will be put in a more convenient form by using the following difference operators.

$$
\begin{align*}
d_{k} & =x-x_{k}  \tag{49}\\
d_{k+1} & =x-x_{k+1} \tag{50}
\end{align*}
$$

Adding and subtracting $X_{k}$ to the right-hand side of equation (50) gives

$$
\begin{equation*}
d_{k+1}=x-x_{k+1}+x_{k}-x_{k}=d_{k}-\Delta x_{k} \tag{51}
\end{equation*}
$$

Substituting equations (49) and (50) in equations (29) and (30) gives

$$
\begin{align*}
y(x) & =M_{k+1} \frac{d_{k}^{3}}{6 \Delta x_{k}}-M_{k} \frac{d_{k+1}^{3}}{6 \Delta x_{k}}+\left(\frac{y_{k+1}}{\Delta x_{k}}-\frac{M_{k+1} \Delta x_{k}}{6}\right) d_{k}-\left(\frac{y_{k}}{\Delta x_{k}}-\frac{M_{k} \Delta x_{k}}{6}\right) d_{k+1} \\
y^{\prime}(x) & =M_{k+1} \frac{d_{k}^{2}}{2 \Delta x_{k}}-M_{k} \frac{d_{k+1}^{2}}{2 \Delta x_{k}}+\left(\frac{y_{k+1}}{\Delta x_{k}}-\frac{M_{k+1} \Delta x_{k}}{6}\right)-\left(\frac{y_{k}}{\Delta x_{k}}-\frac{M_{k} \Delta x_{k}}{6}\right)  \tag{52}\\
& =M_{k+1} \frac{d_{k}^{2}}{2 \Delta x_{k}}-M_{l:} \frac{d_{k+1}^{2}}{2 \Delta x_{k}}+\frac{\Delta y_{k}}{\Delta x_{k}}-\frac{\Delta M_{k} \Delta x_{k}}{6} \tag{53}
\end{align*}
$$

where $\Delta M_{k}=M_{k+1}-M_{k}$.
From equations (49), (50), and (51), when $x=x_{k}$, then $d_{k}=0$ and $d_{k+1}=-\Delta x_{k}$; when $x=x_{k+1}$, then $d_{k}=\Delta x_{k}$ and $d_{k+1}=0$. Substituting these results in equation (53) gives, for $x=x_{k}$,

$$
\begin{align*}
y^{\prime}\left(x_{k}\right) & =-M_{k} \frac{\Delta x_{k}}{2}+\left(\frac{y_{k+1}}{\Delta x_{k}}-\frac{M_{k+1} \Delta x_{k}}{6}\right)-\left(\frac{y_{k}}{\Delta x_{k}}-\frac{M_{k} \Delta x_{k}}{6}\right) \\
& =\frac{\Delta y_{k}}{\Delta x_{k}}-\frac{\Delta x_{k}}{6}\left(M_{k+1}+2 M_{k}\right) \tag{54}
\end{align*}
$$

and, for $x=x_{k+1}$,

$$
\begin{align*}
\mathrm{y}^{\prime}\left(\mathrm{x}_{\mathrm{k}+1}\right) & =\mathrm{M}_{\mathrm{k}+1} \frac{\Delta \mathrm{x}_{\mathrm{k}}}{2}+\left(\frac{\mathrm{y}_{\mathrm{k}+1}}{\Delta \mathrm{x}_{\mathrm{k}}}-\frac{\mathrm{M}_{\mathrm{k}+1} \Delta \mathrm{x}_{\mathrm{k}}}{6}\right)-\left(\frac{\mathrm{y}_{\mathrm{k}}}{\Delta \mathrm{x}_{\mathrm{k}}}-\frac{\mathrm{M}_{\mathrm{k}} \Delta \mathrm{x}_{\mathrm{k}}}{6}\right) \\
& =\frac{\Delta \mathrm{y}_{\mathrm{k}}}{\Delta \mathrm{x}_{\mathrm{k}}}+\frac{\Delta \mathrm{x}_{\mathrm{k}}}{6}\left(\mathrm{M}_{\mathrm{k}}+2 \mathrm{M}_{\mathrm{k}+1}\right) \tag{55}
\end{align*}
$$

The use of these recursive relations is described in the following six cases.
Case 1: Solution of the set of simultaneous equations given by equation (35) when the second derivatives of the function are known at the initial and terminal tabulated boundaries. In this case, $y^{\prime \prime}\left(x_{1}\right)=M_{1}$ and $y^{\prime \prime}\left(x_{n}\right)=M_{n}$. From equation (41) with $k=2$, if $Q_{1}=0$, then $M_{1}=U_{1}$. By using these values for $Q_{1}$ and $U_{1}$, equations (47) and (48) can be used to generate values of $Q_{k}$ and $U_{k}$, recursively, for all values of $k$ ranging from 2 to $n-1$. After these values are available, then all values of $\mathrm{M}_{\mathrm{k}}$ can be computed by using equation (40) written in the form of equation (41)

$$
\begin{equation*}
M_{k}=Q_{k} M_{k+1}+U_{k} \tag{56}
\end{equation*}
$$

and propagating the solution backwards with $k$ ranging from $n-1$ to 2. As $M_{n}$ and $M_{1}$ are known, all $n$ second-derivative values will be available.

Case 2: Solution of the set of simultaneous equations given by equation (35) when the first derivatives of the function are known at the initial and terminal tabulated boundaries. In this case, $y^{\prime}\left(x_{1}\right)=D_{1}$ and $y^{\prime}\left(x_{n}\right)=D_{n}$. Substitute $k=1$ in equation (54) for $M_{k+1}$ from equation (40) and solve for $U_{1}$ to get

$$
\left.\begin{array}{l}
\mathrm{y}^{\prime}\left(\mathrm{x}_{1}\right)=\mathrm{D}_{1}=\frac{\Delta \mathrm{y}_{1}}{\Delta \mathrm{x}_{1}}-\frac{\Delta \mathrm{x}_{1}}{6}\left(\frac{\mathrm{M}_{1}-\mathrm{U}_{1}}{\mathrm{Q}_{1}}+2 \mathrm{M}_{1}\right) \\
\mathrm{U}_{1}=\mathrm{Q}_{1}\left[\mathrm{M}_{1}\left(2+\frac{1}{\mathrm{Q}_{1}}\right)+\frac{6}{\Delta \mathrm{x}_{1}}\left(\mathrm{D}_{1}-\frac{\Delta \mathrm{y}_{1}}{\Delta \mathrm{x}_{1}}\right)\right] \tag{57}
\end{array}\right\}
$$

If $Q_{1}=-1 / 2$, then from equation (57),

$$
\begin{equation*}
\mathrm{U}_{1}=\frac{3}{\Delta \mathrm{x}_{1}}\left(\frac{\Delta \mathrm{y}_{1}}{\Delta \mathrm{x}_{1}}-\mathrm{D}_{1}\right) \tag{58}
\end{equation*}
$$

By using $Q_{1}=-1 / 2$ and the value obtained for $U_{1}$ from equation (58), compute all values of $Q_{k}$ and $U_{k}$ from equations (47) and (48) for values of $k$ ranging from 2 to $n-1$. With the value of $Q_{n-1}$ and $U_{n-1}$ thus computed, substitute
equation (41) in equation (55) for $M_{n-1}$ such that at $x=x_{n}$,

$$
\begin{equation*}
y^{\prime}\left(x_{n}\right)=D_{n}=\frac{\Delta y_{n-1}}{\Delta x_{n-1}}+\frac{\Delta x_{n-1}}{6}\left(Q_{n-1} M_{n-1}+U_{n-1}+2 M_{n}\right) \tag{59}
\end{equation*}
$$

Solving for $M_{n}$ from equation (59) gives

$$
\begin{equation*}
M_{n}=\frac{\frac{6}{\Delta x_{n-1}}\left(D_{n}-\frac{\Delta y_{n-1}}{\Delta x_{n-1}}\right)-U_{n-1}}{2+Q_{n-1}} \tag{60}
\end{equation*}
$$

With the value of $M_{n}$ known, the solution can be propagated backwat $d$ by using equation (41) with $k$ ranging from $n-1$ to 1 . Thus, $M_{1}$ will be computed as the last value.

Case 3: Solution of the set of simultaneous equations given by equation (35) when the first and second derivatives of the function are known, respectively, at the initial and terminal tabulated boundaries. In this case, $y^{\prime}\left(x_{1}\right)=D_{1}$ and $y^{\prime \prime}\left(x_{n}\right)=M_{n}$. Proceed as in case 2 excluding the computation for $M_{n}$, which is not needed.

Case 4: Solution of the set of simultaneous equations given by equation (35) when the first and second derivatives of the function are known, respectively, at the terminal and initial tabulated boundaries. In this case, $y^{\prime}\left(x_{n}\right)=D_{n}$ and $y^{\prime \prime}\left(x_{1}\right)=M_{1}$. Proceed as in case 1 and compute the value of $M_{n}$ as in case 2.

Case 5: Solution of the set of simultaneous equations given by equation (35) when the first and second derivatives of the function are known at the initial tabulated boundary. In this case, $y^{\prime}\left(x_{1}\right)=D_{1}$ and $y^{\prime \prime}\left(x_{1}\right)=M_{1}$. From equation (56), if $Q_{1}=1$, then

$$
\begin{equation*}
\mathrm{U}_{1}=3 \mathrm{M}_{1}+\frac{6}{\Delta \mathrm{x}_{1}}\left(\mathrm{D}_{1}-\frac{\Delta \mathrm{y}_{1}}{\Delta \mathrm{x}_{1}}\right) \tag{61}
\end{equation*}
$$

With the value of $Q_{1}=1$ and the value of $U_{1}$ given by equation (61), compute all values of $Q_{k}$ and $U_{k}$ from equations (47) and (48) for $k=2$ to $n-1$. With these values of $Q_{k}$ and $U_{k}$ known, compute values for the second derivative from equation (40) by propagating the solution forward for $k$ ranging from 1 to $n-1$. The last value thus computed is $M_{n}$.

Case 6: Solution of the set of simultaneous equations given by equation (35) when the first and second derivatives of the function are known at the terminal tabulated boundary. In this case, $y^{\prime}\left(x_{n}\right)=D_{n}$ and $y^{\prime \prime}\left(x_{n}\right)=M_{n}$. In equation (55), substitute $M_{k}$ in equation (41) with $k=n-1$ and solve for $U_{n-1}$ to get

$$
\left.\begin{array}{l}
y^{\prime}\left(x_{n}\right)=D_{n}=\frac{\Delta y_{n-1}}{\Delta x_{n-1}}+\frac{\Delta x_{n-1}}{6}\left(Q_{n-1} M_{n}+U_{n-1}+2 M_{n}\right) \\
U_{n-1}=\frac{6}{\Delta x_{n-1}}\left(D_{n}-\frac{\Delta y_{n-1}}{\Delta x_{n-1}}\right)-M_{n}\left(2+Q_{n-1}\right) \tag{62}
\end{array}\right\}
$$

If $Q_{n-1}=1$, then from equation (62),

$$
\begin{equation*}
U_{n-1}=\frac{6}{\Delta x_{n-1}}\left(D_{n}-\frac{\Delta y_{n-1}}{\Delta x_{n-1}}\right)-3 M_{n} \tag{63}
\end{equation*}
$$

Solving for $Q_{k-1}$ and $U_{k-1}$ from equations (47) and (48) gives

$$
\begin{align*}
& \mathrm{Q}_{\mathrm{k}-1}=-\frac{1}{\Delta \mathrm{x}_{\mathrm{k}-1}}\left[\frac{\Delta \mathrm{x}_{\mathrm{k}}}{\mathrm{Q}_{\mathrm{k}}}+2\left(\Delta \mathrm{x}_{\mathrm{k}}+\Delta \mathrm{x}_{\mathrm{k}-1}\right)\right]=-\frac{1}{\Delta \mathrm{x}_{\mathrm{k}-1}}\left[\frac{\Delta \mathrm{x}_{\mathrm{k}}}{\mathrm{Q}_{\mathrm{k}}}+2\left(\mathrm{x}_{\mathrm{k}+1}-\mathrm{x}_{\mathrm{k}-1}\right)\right]  \tag{64}\\
& \mathrm{U}_{\mathrm{k}-1} \tag{65}
\end{align*}=\frac{1}{\Delta \mathrm{x}_{\mathrm{k}-1}}\left[\mathrm{U}_{\mathrm{k}} \frac{\Delta \mathrm{x}_{\mathrm{k}}}{\mathrm{Q}_{\mathrm{k}}}-6\left(\frac{\Delta \mathrm{y}_{\mathrm{k}-1}}{\Delta \mathrm{x}_{\mathrm{k}-1}}-\frac{\Delta \mathrm{y}_{\mathrm{k}}}{\Delta \mathrm{x}_{\mathrm{k}}}\right)\right] \quad \mathrm{l}
$$

With $Q_{n-1}=1$ and $U_{n-1}$ given by equation (63), compute all values of $Q_{k-1}$ and $U_{k-1}$ from equations (64) and (65) for values of $k$ ranging from $n-1$ to 2. With the values of $Q_{k}$ and $U_{k}$ known, compute values for the second derivatives from equation (40) by propagating the solution backwards for $k$ ranging from $n-1$ to 1 . The last value thus computed will be $M_{1}$.

Clearly, the solutions for case 5 and case 6 could have been obtained from equations (54) and (55). In the solution for case $5, \mathrm{M}_{\mathrm{k}+1}$ can be obtained from equation (54) and $y^{\prime}\left(x_{k+1}\right)$ from equation (55). The solution can be propagated forward recursively with $k$ ranging from 1 to $n-1$. In the solution for case 6 , $M_{k}$ can be obtained from equation (55) and $y^{\prime}\left(x_{k}\right)$ from equation (54). The solution can be propagated backward recursively with $k$ ranging from $n-1$ to 1 .

If equation (52) is written with $d_{k+1}$ replaced by its equivalent $d_{k+1}=d_{k}-\Delta x_{k}$ given ky equation (51), the result is

$$
\begin{align*}
y(x)= & M_{k+1} \frac{d_{k}^{3}}{6 \Delta x_{k}}-M_{k} \frac{\left(d_{k}-\Delta x_{k}\right)^{3}}{6 \Delta x_{k}}+\left(\frac{y_{k+1}}{\Delta x_{k}}-\frac{M_{k+1} \Delta x_{k}}{6}\right) d_{k} \\
& -\left(\frac{y_{k}}{\Delta x_{k}}-\frac{M_{k} \Delta x_{k}}{6}\right)\left(d_{k}-\Delta x_{k}\right)=y_{k}+\left[\frac{\Delta y_{k}}{\Delta x_{k}}-\frac{\Delta x_{k}}{6}\left(M_{k+1}+2 M_{k}\right)\right] d_{k} \\
& +\frac{M_{k}}{2} d_{k}^{2}+\frac{\Delta M_{k}}{6 \Delta x_{k}} d_{k}^{3} \tag{66}
\end{align*}
$$

Equation (66) is the equation for the cubic spline function coefficients in the kth interval. If there are $n$ data points in the tabulation, there will be $n-1$ intervals and $n-1$ cubic spline functions to cover the range oif tabulated data.

For computation purposes, equation (67) is usually written as

$$
\begin{equation*}
y(x)=y_{k}+\left\{\left[c_{3}(k) d_{k}+c_{2}(k)\right] d_{k}+c_{1}(k)\right\} d_{k} \tag{67}
\end{equation*}
$$

where

$$
\begin{align*}
& c_{1}(k)=\frac{\Delta y_{k}}{\Delta x_{k}}-\frac{\Delta x_{k}}{6}\left(M_{k+1}+2 M_{k}\right)  \tag{68}\\
& c_{2}(k)=\frac{M_{k}}{2}  \tag{69}\\
& c_{3}(k)=\frac{\Delta M_{k}}{6 \Delta x_{k}} \tag{70}
\end{align*}
$$




Equations (67) through (70) show that the coefficients are functions of the interval; however, the coefficients need to be computed only once. After the computation is made, the set pertaining to the interval in which the value of the independent variable lies can be brought in to compute the value of the dependent variable needed. Because there are three coefficients for each interval, if there are n data points, there will be $n-1$ intervals and $3(n-1)$ coefficients.

## CONCLUDING REMARKS AND RECOMMENDATIONS

The atmosphere package for the 1962 and 1966 standard atmosphere models is recommended without reservation. When these models were built, it was possible to conduct comparison checks with the layered versions of these models in 250 -meter increments. In both packages, these tests revealed errors that were properly corrected. The package for the 1963 Patrick Air Force Base reference standard model was not subjected to such strenuous tests because the layered version of this model was not available. However, with the other models so thoroughly tested, the choice of altitudes was established as adequate, and the only requirement for the Patrick Air Force Base reference standard model was to ensure that the data were tabulated correctly. The laborious task of comparing (digit for digit) each tabulated number in the model printouts with the numbers in the reference source was performed as carefully as possible, and all errors found were corrected. As a result, the accuracy of this model is reasonably ensured.

To test the accuracy of these models after they have been recoded for use in the Software Development Laboratory, the only requirement is to have the package reproduce the tabulated data. Because the interpolation is to be performed in a different set of units, the output must be converted to the units of the tabulated data for ease of comparison. The results should agree within an error of no greater than $10^{-8}$. Additional verification can be obtained by evaluating the parameters of the models at various altitudes and comparing the results with those obtained from established operational sources, such as the space vehicle dynamic simulator and the Shuttle optimal abort programs. These tests should be performed for all models.

It is recommended that where the first and second derivatives are known at the same data point the cubic spline functions be used to interpolate data only in a small neighborhood of this point and not be extended for interpolation over the entire range of tabulated data.

Lyndon B. Johnson Space Center<br>National Aeronautics and Space Administration<br>Houston, Texas, June 4, 1976<br>986-16-00-00-72

## REFERENCES

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2. United States Committee on Extension to the Standard Atmosphere: U. S. Standard Atmosphere Supplements, 1966. U. S. Government Printing Office, 1966.
3. Smith, O. E.; and Weidner, Don K.: A Reference Atmosphere for Patrick Air Force Base, Florida, Annual (1963 Revision). NASA TM X-53139, 1964.


Figure 1.- Undeformed beam.


Figure 2.- Deformed section of a beam.


Figure 3.- Element of the cross section of a beam.


Figure 4.- Graph of a tabulated set of data for cubic spline interpolation.


APPENDIX
PROGRAM FLOW CHARTS AND LISTINGS

Defintions of the symbols used in the program flow charts and listings are as follows.

## SYMBOLS USED IN SUBROUTINE SDAT (IOP)

IOP an integer parameter used to designate the desired atmosphere based on the following set of values:

IOP $=0 \quad 1962$ standard atmosphere
IOP = 11966 standard atmosphere for July at latitude $30^{\circ} \mathrm{N}$
$\mathrm{IOP}=21966$ standard atmosphere for January at latitude $30^{\circ} \mathrm{N}$
$10 P=31966$ standard atmosphere for July at latitude $60^{\circ} \mathrm{N}$
IOP $=41966$ standard atmosphere for January at latitude $60^{\circ} \mathrm{N}$
IOP = 51963 Patrick Air Force Base reference standard atmosphere
NF an integer variable used to specify the number of data points used to describe each atmospheric parameter in each model (equal to 123)

NP64 an integer variable used to specify the total number of data points used to describe all atmospheric parameters in each atmosphere model (equal to $6 * \mathrm{NP}$ or $6 \times 123=738$ )

NP34 an integer variable used to specify the total number of cubic spline coefficients for each atmospheric parameier (equal to $3 *(N P-1)$ or $3 *(123-1)=366$ )

PRSLST a column vector array dimensioned by NP64 or 738 cuntaining all pressure data values (in millibars) for all atmosphere models tabulated

RHOLST a column vector array dimensioned by NP64 or 738 containing all density data values (in kilograms per cubic meter) for all atmosphere models tabulated

SONLST a column vector array dimensioned by NP64 or 738 containing all speed of sound data 1 tes (in meters per second) for all atmosphere models tablilated
$\left.\begin{array}{ll}\text { VISLST } & \begin{array}{c}\text { a column vector array dimensioned by NP64 or } 738 \text { containing all coeffi- } \\ \text { cient of viscosity data values (in kilograms per meter per second) } \\ \text { for all atmosphere models tabulated }\end{array} \\ \text { ALTLST } & \begin{array}{l}\text { a column vector array dimensioned by NP or } 123 \text { containing the altitude } \\ \text { data values (in meters) for all atmosphere models tainulated }\end{array} \\ \text { WORK } & \begin{array}{l}\text { a column vector array dimensioned by NP or } 123 \text { used as a working } \\ \text { array for subroutine SPLN }\end{array} \\ \text { D a column vector array dimensioned by } 2 \text { containing the values of the } \\ \text { first (or second) derivative of the function at either or both of the } \\ \text { tabulated terminal boundaries (only first-derivative values are used } \\ \text { for the atmosphere package) }\end{array}\right\}$

CV a column vector array dimensioned by NP34 or 366 containing the cubic spline function coefficients for the interpolation for the coefficient of viscosity in pounds-mass seconds per square foot

J1 an integer variable used to designate the order of the derivative of the function supplied at the initial tabulated boundary (except when $\mathrm{J} 1=4$ ). It can have the following values:
$\mathrm{J} 1=1$ The first derivative of the function at the first tabulated point is supplied in $\mathrm{D}(1)$.
$\mathrm{J} 1=2$ The second derivative of the function at the first tabulated point is supplied in $D(1)$.
$\mathrm{J}=3$ The first and second derivatives of the function at the first tabulated point are supplied in $\mathrm{D}(1)$ and $\mathrm{D}(2)$, respectively.
$\mathrm{J} 1=4$ The first and second derivatives of the function at the last tabulated point are supplied in $D(1)$ and $D(2)$, respentively.

JN an integer variable used to designate the order of the derivatives of the function supplied at the last tabulated boundary. It can have the following values:
$\mathrm{JN}=1$ The first derivative of the function at the last tabulated pcint is supplied in $\mathrm{D}(2)$.
$\mathrm{JN}=2$ The second derivative of the function at the last tabulated point is supplied in $\mathrm{D}(2)$.
xs, ys1, Variables used for temporary storage in the ordering process for ys2, ys3,- altitude, pressure, density, speed of sound, and coefficient of ys4 viscosity, respectively

SYMBOLS USED IN SUBROUTINE SPLN1(N,X,Y,J1,JN,D,C,W)
$\mathrm{N} \quad$ an integer variable used to specify the number of data points used to represent the function in the region of definition

X
a column vector array dimensioned by at least N in the calling element containing the values of the indepenuent variable in ascending order

Y a column vector array dimensioned by at least N in the calling element containing the values of the dependent variable in styuence with the values in the X array

J1
same as J 1 defined in SDAT

D

C a column vector array dimensioned by at least $3^{*}(\mathrm{~N}-1)$ in the calling element containing the cubic spline function coefficients for the first-, second-, and third-degree terms for each interval. A portion of the array is also used to compute and store the values of one of the recursive functions used in the computation of the second derivatives and in the computation of the second-derivative values.

W a column vector array dimensioned by at least N in the calling element used as a working array to compute and store the values of one of the recursive functions used in the computation of the second derivatives and in the computation of the cubic spline function coefficients

SYMBOLS USED IN SUBROUTINE ATMSPL(V,FANS)

FANS a column vector array dimensioned by 8 containing the following:
FANS (1) the interpolated value for pressure at $V$ altitude in pounds-force per square foot

FANS (2) the interpolated value for density at $V$ altitude in slugs per cubic foot

FANS (3) the interpolated value for speed of sound at V altitude in feet per second

FANS (4) the interpolated value for cotificient of viscosity at $V$ altitude in pounds-mass seconds per square foot

FANS (5) the ratio $c^{*}$ the pressure at $V$ altitude and the pressure at 0 altitude

FANS (6) the ratio of the density at V altitude and the density at 0 altitude

FANS (7) the ratio of the speed of sound at $V$ altitude and the speed of sound at 0 altitude

FANS (8) the ratio of the coefficient of viscosity at $V$ altit :de and the coefficient of viscosity at 0 altitude

All other variables are defined in subroutine SDAT.


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$$
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$$

| $C$ | COMPUTE CUBIC SPLINE INTERPOLATION COEFFICIENTS FOR COEFFICIEAIT OF |
| :--- | :--- |
| $C$ |  |

CALL SPLNI(NP.ALTTAB.VISTAB.JI.JN.D.CV. WORK)
RETURN
END

EG 4 FINAL

## SUBROUTINE SPLNI(N.X.Y.JI.JN.D.C.W)

## B




CONT. ON PG 2
SPLNI
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CONT. ON PG 3 $\qquad$
ATMSPL
PG: OF 4

 FANS (8) $=$ FANS (4)/VISTAB(1) $\frac{1}{\text { RETURN }}$


RETURN
C TERMINAL BOUNDARY VALUES - FUNCTION/SEA LEVEL RATIO


FANS(1) = PRSTAB(NP)
FANS(2) $=$ RHOTRB(NP)
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ATMSPL
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ATMSPL
PG 4 FINAL


























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| $1.281 E-1$. | $9.576 E-2$. | $7.075 E-2$. | $5.160 E-2$. | $3.710 E-2$. | $2.627 E-2$. |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |



|  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 5 |  |  |  |  |  |  |
| 1.310E-4, 9.889E-5. 7.633E-5, 6.004E-5. 4.801E-5. |  |  |  |  |  |  |
| T |  |  |  |  |  |  |
|  | C | 79 | 80 |  | 114 |  |
| 3.903E-5. 3.224E-5. 34*2.700E-51 |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
| DATA (PRSLST(1).1 $=$ NP43.NP44)/ |  |  |  |  |  |  |
| C 115 116 117 |  |  |  |  |  |  |
|  |  |  |  |  |  |  |



PG 27 OF 52



























F.: spans




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| 09101 | 18 |  | Sutreutmit SFLmimer, |
| :---: | :---: | :---: | :---: |
| 2018 | $1 \cdot$ | $c$ |  |
| ens | J• |  |  |
| 9010) | ** | S |  |
| 90101 | ** | c |  |
| 04103 | ** | c |  |
| 0)103 | 10 | c |  |
| 09103 | $\because$ | C |  |
| 90103 | * | C |  |
| 09103 | 190 | C |  |
| 00103 | 11. | c |  |
| 00101 | 120 | c |  |
| 00102 | [30 | c |  |
| 09103 | 19. | c |  |
| 0610) | 160 | c |  |
| 00103 | 16* | c | di,dm - imitcems bifimeo as relleas |
| $0 \cdot 103$ | 110 | c |  |
| 00102 | 10* | C | at - - Finst otetyativt or ine rumction at the rimsp pagulatio |
| cosel | 190 | 5 | Date Potwi is suppileb ic oill |
| 00191 | $80 \%$ | , |  |
| 00:03 | 210 | c |  |
| 00103 | 120 | 6 | oata polmi is supfitio in oill |
| 0ens | 230 | c |  |
| esos | 240 | c |  |
| 0010\% | 25. | $t$ |  |
| $\begin{aligned} & \text { epio) } \\ & \text { colus } \end{aligned}$ | $\begin{aligned} & \text { 20: } \\ & 2 \% \end{aligned}$ | c |  |
| 00103 | 200 | 6 |  |
| 00103 | 200 | c |  |
| 00103 | $30 \cdot$ | 6 |  |
| 60103 | 小* | 6 | Wata Pjims is suertico in oly |



DRIGINAL PAB: OF POOR QUSII:






|  |  |  | $\begin{gathered} r-35<0^{\circ} z \\ r e 1 \end{gathered}$ | $\begin{gathered} 18-3<3_{n}-9 \\ 8<1 \end{gathered}$ | $\begin{gathered} r-3 c \geq n+a x \\ 121 \end{gathered}$ | ， | $\begin{aligned} & . \operatorname{Haz} \\ & . H 92 \end{aligned}$ | $\begin{aligned} & 0 \times 100 \\ & 0 n 100 \end{aligned}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| －1－39m1．9 | －1－3m 2 A •t | $137800 \cdot 1$ | 138mor | －2760r．2 | －232n5－ry |  | ． 192 | 0.100 |
| 021 | 6：1 | Q1！ | 611 | －： 1 | ¢！ 1 | ， | －99？ | S¢ 100 |
|  |  |  |  |  |  |  | － 592 | CH100 |
|  |  |  |  |  |  | ） | － 092 | －1 190 |
|  |  |  |  | ¢ 5－3unter | －9－7550．ty |  | － 602 | 9 1100 |
|  |  |  |  | $\mathrm{O}_{8}$ | 66 | ） | －202 | －CiOC |
| ＇s－3scion | －5－394\％．9 | －9－3mal．a | －$n-759001$ | －¢－7く26－1 | 2Ansentilx |  | .192 | － 1100 |
| 96 | 46 |  | 54 |  | ： 6 | ， | .092 | －1100 |
| －$n-3.5102$ | ＇n－3bcaic | －n＊sancs | － $\mathrm{m}-3 / 1 \theta^{\circ} \mathrm{C}$ | 18．34n） 1 |  |  | ． 652 | － 1100 |
| 26 | 16 | 01 | 49 | 04 | ${ }^{\circ}$ | 3 | －¢st | － 1100 |
| ${ }^{\prime} \varepsilon-3 \sin m^{\circ} 2$ | $\begin{gathered} 18-310 \cdot 0 \cdot \varepsilon \\ 59 \end{gathered}$ | $\underset{\text { He }}{6-7: 0 r}$ | $\begin{gathered} 1-3231^{\circ} L \\ 59 \end{gathered}$ |  | $\begin{gathered} 2-362001 x \\ 19 \end{gathered}$ | ， | $.952$ | $\begin{aligned} & \text { - } 1100 \\ & +1100 \end{aligned}$ |
| －z－390¢－z | －2－7n2205 | －2－759m－i， | －Z－3「80＊ |  | 11－3201．1x |  | － 552 | －r100 |
| 09 | 65 | \％ 5 | c 5 | 95 | 55 |  | －＊52 | －riou |
| －1－329n－1 | －1－3．26．1 | －1－7215．2 | 11．3652 ${ }^{\circ}$ | －1－3212．＊ | 11－3n2r－sx |  | －\｛5 5 | － 100 |
| ＊ 5 | ［ 3 | 23 | 15 | $\mathrm{O}_{5}$ | 6 ＊ |  |  | － 1100 |
| － $1-3656^{\circ}$ | －1－3806．8 | 4070n1．1 | －0314\％ | －07180．1 | －03ctmezx | ， | .252 .152 | － 1100 |
| \％＇ | C＂ | $9{ }^{\text {a }}$ | 5 h | Hn | ¢＂ | ， | .052 | － 1100 |
| －03c91．8 | －03＇11．4 | －03c2n－s | － $03591{ }^{\text {c }}$ |  | －13cerelx |  | －6mz | － 5100 |
| 2\％ | 1 h | Or | 6 t | ${ }^{8} \mathrm{r}$ | LT | J | －8， 8 | 9¢100 |
| －1350く．1 | －1300r．z | －17121．c | －13852\％ | －179ヶpに | －13＜9808x |  | $\begin{aligned} & \operatorname{ein} 2 \\ & \sin 2 \end{aligned}$ | － 1100 |
| 9 c | 5 C | H C | ¢ $¢$ | 2 C | if | 2 |  | － 1100 |
| －13180．8 | 11352506 | －23521．1 | －－atar＊ | －23585．1 | －239「日－1x | 3 | $\sin 2$ | －1 100 |
| Of | 62 | 82 | 12 | 92 | 52 | ， | －nt2 | － 1100 |
| －23191－2 | －2300＊－2 | － $235<88.2$ | $123 \angle O E{ }^{\circ}$ | $.27186 \cdot 8$ | －231ctonx |  | $\begin{aligned} & r_{n} \\ & \cdot n^{2} 2 \end{aligned}$ | －rino |
| M2 | ¢ 2 | 22 | $12$ | $\mathrm{O}_{2}$ | 61 | ， |  | －1800 |
| $\begin{gathered} 23.20^{\circ} \mathrm{m} \\ 81 \end{gathered}$ | $\begin{gathered} 23 \mathrm{~g} 5 \cdot 5 \\ 6 \end{gathered}$ | $\begin{gathered} 23850^{\circ} 9 \\ 91 \end{gathered}$ | $\begin{gathered} 2318 t^{\circ} 9 \\ 51 \end{gathered}$ | $=2 \div 0 z<0$ |  |  | －imz | $\begin{array}{r} 70 \\ \square \\ i \end{array}$ |
| $\begin{gathered} 23590^{\circ} \\ 21 \end{gathered}$ | $\begin{gathered} 23289.2 \\ 11 \end{gathered}$ | $123: 10.4$ | $.23150^{\circ} 8$ | $423^{1}+2 \cdot \theta$ | $\cdot 230 \mathrm{~g} \cdot \mathrm{\theta}$ | 3 | $\begin{aligned} & \text { obrz } \\ & \text { - vcz } \end{aligned}$ | －r 100 |
| 1230919 | －Eこ甲以の－0 | －23016．t | ． $23685^{\circ}$ 。 | 123nçe．4 | － 30 scio． $1 x$ |  |  | ¢ 1100 － 100 |
| － | 5 | － | ¢ | 8 | 1 | 3 | $\begin{aligned} & \text {-ict } \\ & -5 r z \end{aligned}$ | －rior |
|  |  |  | ／122dN＇t | $\mathrm{dN}=1 \cdot 1: 125$ | 758d）vivo |  |  | － 1 100 |
|  |  |  |  |  |  | ， | －he？ | ¢ 5100 |
|  | ¢ูvア！ |  |  |  |  | ， | $\begin{aligned} & \operatorname{ccz} \\ & -2 c z \end{aligned}$ | $\begin{aligned} & m 「 100 \\ & h r 100 \end{aligned}$ |
|  |  |  |  |  |  |  |  |  |
| 30＾1：177 | HAEON S334930－0 AV47 |  | 7nesous 3 ym | 4504！O8POnvis 9901 |  | j | $\begin{aligned} & -1 c z \\ & -0 c z \\ & -0, \end{aligned}$ | $\begin{aligned} & \text { Mr100 } \\ & \text { MC100 } \end{aligned}$ |
|  |  |  |  |  |  |  |  |  |
|  |  |  | .622 |  |  | －croo |  |  |
|  |  |  | 15－301z．t－r $\quad$ \％ | 3 | －822 |  | $\rightarrow C 100$ |  |
|  |  |  | czi 21 | ， | $\begin{aligned} & -122 \\ & .922 \end{aligned}$ |  |  |  |
| 5－3186＊ 1 | 19．31504．1．2 |  |  |  |  | －9－3915n＊ |  |  | HC100 |
| 021 | $s 11$－ | 811 |  | 11 | $\bullet 11$ | 511 | ， | 0.922 .522 | W5100 |
|  |  |  |  |  |  |  |  | － 6100 |
|  |  |  |  |  |  | ， | $\begin{aligned} & 1222 \\ & 0 \\ & 122 \\ & 122 \end{aligned}$ | $\begin{aligned} & 2 ¢ 100 \\ & 26100 \\ & 28100 \end{aligned}$ |
|  |  |  |  |  | $\text { - 5-3ze2. } 14$ | ， |  |  |
| －5－3 0 OCP 1 | －5－35！5c．l |  |  | －9－JPRCm－1 | －5－7909hol | 19－30225．1x |  | － 022 | 21100 |
| $0 \cdot$ | 050 | 85 | ＜s | －9 | 55 | ， | －612 | 21100 |
| －5－3yi95．1 |  |  | $\begin{gathered} +G-7 \angle R * O^{\circ} 1 \\ 1 \leq \end{gathered}$ | $\begin{gathered} -5-39890 \cdot 1 \\ 05 \end{gathered}$ | －5－76089－1x | ， | －Wlz | Y 110025100 |
| ns |  |  | $6 \%$ |  | －112 |  |  |  |
|  | 25－3：506－1．0 |  |  | $\begin{gathered} 5-3 \operatorname{lc} 80^{\circ} 1 \\ 50^{\circ} \end{gathered}$ | －5－7rcso．1 |  | －5－75¢で・1x | －912 | $28100$ |
|  | 日＊－－${ }^{\text {a }}$ |  | ＊${ }^{\text {n }}$ |  | 「＊ | ， | － 512 | 25100 |
| 5－3600．01 | －5－3t2＜5－1 | －cearchs．l | －9－7cnis＊） | －5．765日＊－i | －¢－3C5c＊＊ $1 \times$ |  | － 12 | C100 |
| th | $1 \times$ | 3n | as | $\square^{\circ}$ | A | ， | －112 | ef100 |
| －5－3＊non－1 |  |  |  |  |  |  |  |  |


| 00: 0 | 270 | 5 |
| :---: | :---: | :---: |
| -01+0 | $271^{\circ}$ | $C$ |
| $001 * 0$ | $171^{\circ}$ | $C$ |
| 00140 | 2710 | $c$ |
| $001+2$ | 270 |  |
| 001.1 | 215* | $c$ |
| 001.2 | $276^{\circ}$ |  |
| 091.2 | 277* | $C$ |
| 00192 | 27 m |  |
| 001-2 | $270{ }^{\circ}$ | $C$ |
| 00142 | $20^{\circ}$ |  |
| 00142 | $281{ }^{\circ}$ | $C$ |
| 00141 | 26* |  |
| $0014:$ | 20] | $C$ |
| $00192$ | $26^{\circ}$ |  |
| 00192 | 205 | $C$ |
| 00192 | $26^{\circ}$ |  |
| 00142 | 467* | C |
| 30192 | 2en |  |
| cota 2 | 210* | $C$ |
| 101*2 | $2^{\circ} 0^{\circ}$ |  |
| 00142 | 20, | C |
| 00152 | 2* $2^{\circ}$ |  |
| 00152 | 291* | C |
| 00172 | 2740 |  |
| 00142 | 24* | C |
| 05142 | 276 |  |
| 00142 | 297 | C |
| cold | 298 |  |
| 0014? | 200* | 6 |
| 00142 | $300^{\circ}$ |  |
| 00142 | $301{ }^{\circ}$ | C |
| 00142 | $302{ }^{\circ}$. |  |
| 00142 | 101* | $C$ |
| 00149 | 10** |  |
| 00154 | $305^{\circ}$ | $C$ |
| 00144 | $306^{\circ}$ |  |
| 00149 | $307{ }^{\circ}$ | $C$ |
| 00149 | $300^{\circ}$ |  |
| 00149 | $300^{\circ}$ | C |
| 0019* | $310^{\circ}$ | 6 |
| 00144 | 3110 | c |
| 00174 | $312^{\circ}$ | 6 |
| 0014 4 | 3130 |  |
| 0014 | 310* | $C$ |
| 00146 | $315^{\circ}$ |  |
| 001*6 | $314^{\circ}$ | C |
| 00176 | $317^{\circ}$ |  |
| $0 \mathrm{c} 1+6$ | 319* | C |
| 00146 | $319{ }^{\circ}$ |  |
| 00146 | $320^{\circ}$ | C |
| 00146 | 321* |  |
| 001*6 | 122* | C |
| 0014* | 323* |  |
| 00146 | 324* | C |
| 00146 | 125* |  |
| 0014* | 126* | C |
| 00146 | 321* |  |







| 00174 | $500^{\circ}$ |  |  | 6.57162. | $6.164520$ | $5.779 \pm 20$ | $5 \cdot 714 \in 2$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $00: *$ | 54， | C | 17 | 20 ${ }^{2}$ | $21$ | $22$ | $23$ | $29$ |
| 00176 | S＊2＊ |  | 14．134E2． | 3－5 ${ }^{\circ} \mathrm{EL}$ | 3.1642. | 2－677E2． | 2．301E2． | $1.978 E 2,$ |
| 00176 | 363＊ | 6 | 25 | ${ }_{4}$ | 21 | 2\％ | 2＊ | 30 |
| 00176 | 544＊ |  | $\times 1.700 \mathrm{E}$ ． | 1－40：と2． | 1．256E2． | 1．08Jez， | －．205EI． | 7－73） |
| 00170 | 565＊ | C | 11 | 32 | 33 | 34 | 35 | 16 |
| 00170 | 546＊ |  | X0．004EI， | 5－9J2Et， | －．364E1． | 3.229610 | 2－30日E1． | 1．788EI． |
| cosi 70 | 54\％ | C | 3） | 3 s | 39 | $\rightarrow 0$ | －1 | 42 |
| 00176 | 508＊ |  | $21.33^{6} 10$ | 10cこちを1＇ | 7．503E06 | S．77300． | $4 \cdot 21 E 0 \cdot$ | 3．4CtEO． |
| 00178 | $50^{\circ}$ | C | $4$ |  | $45$ |  | $0.072(10$ | 7.142c-1. |
| 00176 | $570^{\circ}$ |  | 120040E2： | $2 \cdot 05 ; 60^{\circ}$ | $\begin{gathered} 1.607600 \\ 51 \end{gathered}$ | $\begin{gathered} 1.250<00 \\ 52 \end{gathered}$ | $\begin{gathered} 9.072 E=10 \\ 53 \end{gathered}$ | $\begin{gathered} 7.142 E^{-10} \\ 59 \end{gathered}$ |
| 0017\％ | 5710 | 6 |  |  | 3．70¢E－1． | 2－811E－1． | $2 \cdot 2^{53}$ | $1.69+E-1$ |
| 00176 <br> 00176 | $573$ | C | $\begin{gathered} 55 \\ \times 1.281 E-1 . \end{gathered}$ | $9.5_{B E}^{56}$ | $\begin{gathered} 5 y \\ 7.0756=20 \end{gathered}$ | $\begin{gathered} 59 \\ 5 \cdot 100 \mathrm{E}=2 . \end{gathered}$ | $\begin{gathered} 59 \\ 1.710 \mathrm{E}-2, \end{gathered}$ | $2.627 E=2 .$ |
| $\begin{aligned} & 00176 \\ & 00176 \end{aligned}$ | $\begin{aligned} & 575 \\ & 570^{\circ} \end{aligned}$ | C | $11 \cdot 0^{i t} \varepsilon=2:$ | $1 \cdot 2^{\Delta \omega_{i}} \mathrm{t}-2 .$ | $6.34 \mathrm{OE}_{3} .$ | $5.530 t-3 .$ | $3 . \mathscr{4}_{7_{E}}$ | $2.430 E=3 .$ |
| 00176 | 57\％ | C | $\epsilon^{\prime}$ | ${ }^{2} 8$ | 69 | 70 |  | 12 |
| coit ${ }^{\text {c }}$ | 5780 |  | 11－6：CE－3． | 10c：5t－3． | 7．29：6－4． | S－C27E－4． | 303．2E－4． | 2．485E－4． |
| 001： 0 | $579{ }^{\circ}$ | C | 13 |  | 75 | 74 | 77 | 18 |
| 00170 | $50^{\circ}$ |  | 1．188204， | 1－310ヶ＊＊ | 9.809 E － | 7．633E－5， | $6.004 E-5$. | －001E－S． |
| 00176 | $581{ }^{\circ}$ | C | 79 | ${ }^{3} 0$ |  | $1-114$ |  |  |
| 00176 | 50， |  | 13．903E－3． | J．2z－E－5． |  | 2．700E－5／ |  |  |
| 00176 | 583＊ | c |  |  |  |  |  |  |
| 00200 | 56＊＊ |  | data iprsi | cril．1＝nip |  |  |  |  |
| 00200 | $585 *$ | C | 115 | $: 0$ | $\mathrm{il}^{7}$ | 110 | 110 | 120 |
| 00200 | $584^{\circ}$ |  | 83．343E2． | 2－13うE2• | 3.75361. | 1－1isto， | 8．742E－1． | $6.857 E=1$. |
| 00200 | $567^{\circ}$ | C | 121 | 122 | 123 |  |  |  |
| 00200 | $580^{\circ}$ |  | 11－u25E－2， | 6．80） $0^{-3 .}$ | $1.978 \mathrm{EC3/}$ |  |  |  |
| 00260 | $589^{\circ}$ | $c$ |  |  |  |  |  |  |
| 00200 | $50^{\circ}$ | C |  |  |  |  |  |  |
| 00200 | 5910 | c | OENSI | Tr blucs | a日ibateo I | KILOGMAMS | PER CUBIC | PER |
| 00200 | 5\％2＊ | $C$ |  |  |  |  |  |  |
| 00202 | $53^{\circ}$ |  | Data inmol | S11：1，10ヶp | －4p＋211 |  |  |  |
| 00202 | $5 \% 4 *$ | $c$ | 1 | 2 | 3 | 4 | 5 | $\bigcirc$ |
| co202 | $5 \%{ }^{\circ}$ |  | 11．22000． | 1．190く0． | 1.16160. | 1－132201 | 1－109EO． | 1－077E0． |
| 00202 | $59{ }^{\circ}$ | $C$ | 7 | － | 9 | 10 | 11 | 12 |
| 00292 | $5 \%^{\circ}$ |  | 21．050es． | 1．023E」＊ | －．97E－1． | 9．710E－1． | 9．467E－1． | － 223 ¢ 1 ， |
| 00202 | $598{ }^{\circ}$ | C | 13 | $!$ | is | 10 | $!7$ | 14 |
| 00202 | $599^{\circ}$ |  | 58．984E－1． | 2．52it－1． | 8．077E－1． | 7－6516－1． | 7．244E－10 | 6．519E－1． |
| 00202 | $600^{\circ}$ | C | 19 | 2 C | 21 | 22 | 23 | 24 |
| 00202 | $601{ }^{\circ}$ |  | KS．E49E－！ | S．23ik－！ | $4.063 \mathrm{H}-1$. | －－142E－1． | 3．560E－1． | 3－0408－1． |
| 00202 | $602^{\circ}$ | C | 25 | 26 | 27 | 28 | 29 | 30 |
| 00202 | $\bigcirc 03^{\circ}$ |  | ¢2，03IL－1． | 2．20： | 1.944 ECO | 1．6716－1． | 1－43E－1． | 1．235E－1， |
| $\begin{aligned} & 00202 \\ & 00202 \end{aligned}$ | $\begin{aligned} & 604^{\circ} \\ & 805^{\circ} \end{aligned}$ | C | 11．062E＝1． | $9 \cdot:_{3}^{2} 2_{2}^{2}-2 .$ |  | $4.98_{5}^{34}-2 .$ | $\begin{gathered} 35 \\ 3.03^{9} \mathrm{E}-2 . \end{gathered}$ | $\begin{gathered} 10 \\ 2.65-2 . \end{gathered}$ |
| 00202 | $106^{\circ}$ | （ | 3） | נ明 | 39 | ＊ 0 | 4 | 42 |
| 00202 | $60^{\circ}$ |  | 11．979E－2， | 1．068t－2． | 1．08JE－2． | 8．041E－3． | 6．013E－3， | －．528E－3． |
| 00202 | －0， | 6 | 4 | －＂ | 45 | $4{ }^{4}$ | 47 | 48 |
| 00202 | －09＊ |  | 33．4JIE－J． | z－Eうge－3． | 2．039－ 3. | 1－5d4t－3． | 1．241E－3． | 9．731E－4， |
| 00202 | 4100 | 6 | 49 | 2 | 51 | 32 | 53 | 54 |
| 00202 | －110 |  | 17．007E－4． | 6．Edec－ | 4．814E－4． | 3．815k－4． | 3．049E－4． | 2．416E－4． |
| 00292 | $412^{\circ}$ | $c$ | 55 | $\zeta_{\text {c }}$ | $5)$ | 58 | $5 \%$ | $\triangle 0$ |
| $002: 2$ | $613^{\circ}$ |  | K1．098t－4， | 1－以「7t－4． | 1．13EE－4． | 6．000t－S． | －．521E．5． | － 8 － 2 E－5， |
| 00202 | $614^{\circ}$ | 6 | －！ | 37 | 6） | ${ }^{*}$ | 65 | 40 |
| 00202 | ＊15＊ |  |  | 2•5ン1t－b． | 1．${ }^{\text {coct－5 }}$ ． | 1－102L－3． | 1．30tE－6． | 5．2JJE－6． |
| 00202 | －18＊ | 6 | $6^{\prime}$ | 3 F | 69 | 10 | 31 | 72 |
| 00202 | ＊17＊ |  | 13．40HE－6． | 2．2290－0， | ：．＂5veror | 4．64CE－7． | ＊．491E－7． | － $434 \mathrm{E}-1$. |



| 00212 | 076 |  |  | 1＊ブさにーづ |  | 1．72＊t－5． | 1－121E－5． | 1－115E－S． |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 00212 | 67\％ | 6 | 13 | 14 | 13 | 1. | 17 | 18 |
| 00212 | －7ヵ＊ |  | 11．7csic5． | 1－s9－E－5． | 1．68ut－5． | 1．64t－5． | 1．0528－5． | 1．610c－5， |
| 00212 | －79＊ | $c$ | $1^{*}$ | ${ }^{2} \mathrm{C}$ | ： |  |  |  |
| 00212 | $6^{6} 9^{\circ}$ |  | 21－3＊OE－5． | 1．543t－3． | 1．500E－5， |  |  |  |
| 00212 | ${ }^{61} 1^{\prime}$ | $C$ |  |  | 22. |  |  |  |
| 60212 | $62^{\circ}$ |  | $\pi$ |  | $18 * 1.4$ | 4－5． |  |  |
| 00212 | ＊）〕＊ | $c$ |  |  |  | 34 | 35 | 13 |
| 00212 | $6^{6} 4^{*}$ |  | ！ |  |  | 1－4 70t－5． | 1．4＊2E－5． | 1．5cyes， |
|  | $085$ | $c$ | $37$ |  |  |  |  |  |
| $00212$ | $68{ }^{\circ}$ |  | $x: \operatorname{cose} 5$ | $1.530 E=5 .$ | $1.50+1-5 .$ | $1 . \operatorname{coj}^{2}-5 .$ | $1.03 \text { ic } 5 \text {. }$ | $1.00 i c-5 .$ |
| 00212 | ${ }^{87}{ }^{\circ}$ | $C$ | － 1 | ＊ | 45 | 46 |  | － 48 |
| 00212 | ${ }^{88}{ }^{\text {a }}$ |  | 21－690－5． | 1．713E＝3． | 1．723t－51 | 1．734k－3． | $2 \cdot$ | 36E．3． |
| 00212 | $\bullet^{89}{ }^{\circ}$ | C | 49 | $5_{0}$ | 31 | 52 | 53 | 54 |
| 02212 | $60^{\circ}$ |  | 41．7308－5， | 1．710 $0^{2}-3$. | 1．69 $1^{1}$－5． | 1．06－E－5． | 1．417E－5． | 1．549E－5． |
| 00212 | $\bullet^{9} 1^{\circ}$ | $C$ | 55 | $S_{4}$ | 57 | 58 | $5{ }^{\circ}$ | － 0 |
| 00212 | $692^{\circ}$ |  | 21．321E＝3． | 1＊＊？ $2^{\text {E－S }}$ ． | 1．421E－5． | 1．310t－5． | 1－110E－5． | 1－2．5E－5． |
| 00212 | 6910 | C | 61 | $0^{6}$ |  | 43 | 67 |  |
| 00212 | $6{ }^{\circ}{ }^{\circ}$ |  | －1．21：E－5． | 1．156E－5． |  | $s$ ¢ 1. | 2E＝5 |  |
| 00212 | $695^{\circ}$ | C |  | $6_{6} 8$ | $6{ }^{\circ}$ | 12 | 71 | 72 |
| 00212 | $69{ }^{\circ}$ |  | $x$ | $1-: 4 t^{t-5}$ | 1．182E－5． | 1．222t－3． | 1．262E－5． | 1－202E－5． |
| 00212 | 6970 | C | 73 | 74 | 15 | 76 | 77 | 78 |
| 00212 | $\bullet^{9}$ ค＊ |  | 11．360E－50 | 1．4＇cE－5． | 1．570E－5， | 1－079t－51 | 1．7776－5． | 1．875E－S． |
| 00212 | 699 | C | 79 | ${ }^{8} 0$ |  | 1－114 |  |  |
| 00212 | 700. |  | X1．995E－5． | 2．111E－S． |  | －2．22ik－5 |  |  |
| 00212 | $701^{\circ}$ | C |  |  |  |  |  |  |
| $002{ }^{4}$ | $702^{\circ}$ |  | OATA IVISL | T11：10NPC | （1） $\mathrm{N}^{(4)}$ |  |  |  |
| 00214 | $703^{\circ}$ | C | 115 | $110=$ |  | 11 | － 120 |  |
| 00214 | $704 *$ |  | E1－S24E－5． | $2 \cdot 1$. | $t^{-5}$ |  | ． 736 E－5． |  |
| 0021 | $705^{\circ}$ | $C$ | 121 | 122 － |  |  |  |  |
| 00219 | $700^{\circ}$ |  | 11．129E－5． | 2－1．1 | 2F－5／ |  |  |  |
| 90215 | $70 \%$ | $c$ |  |  |  |  |  |  |
| 40214 | $70{ }^{\circ}$ | c |  |  |  |  |  |  |
| 00214 | 709 | $C$ |  | 1966 | TAmDAD at | OSPMEHE FO |  |  |
| 00219 | 7：00 | C |  | JANUAHY | AT $60-2 E G 4$ | ES NOWTM | ITUOE |  |
| 00214 | 1110 | C |  |  |  |  |  |  |
| 0021 | 7i ${ }^{\circ}$ | $C$ |  | ESSURE Val | UEs tabula | EO INMILL | A9S |  |
| 00214 | $713^{\circ}$ | 6 |  |  |  |  |  |  |
| 00210 | 714＊ |  | Data IPRSL | Y1：1．1ENPS | ，mPS21／ |  |  |  |
| 002： | 7：5＊ | 6 | 1 | 2 | 3 | 4 | 5 | 6 |
| 0021＊ | $710^{\circ}$ |  | 81．01350E3． | －004E2． | 9．－84E2． | 9．176E2． | 0．37ecz． | 6．590\％2： |
| 00216 | 717 | C | 7 | $\bullet$ | － | 10 | 11 | 12 |
| 00216 | $718^{\circ}$ |  | 80．310E2． | 8－03822． | 7．172も2． | 7－519t2＊ | 1．2tit2． | 8．031E2． |
| UJ21＊ | $71{ }^{\circ}$ | C | 13 | 14 | 15 | 16 | 17 | 10 |
| 00216 | $720^{\circ}$ |  | 16．794E2， | 6．3）262： | S．932E2， | S．534t20 | S．158E2． | 4．46）E2， |
| 00210 | 121＊ | 6 | 19 | 20 | 21 | 22 | 23 | 2＊ |
| $0021 *$ | $722^{\circ}$ |  | 23．053E2： | コ・こごき2＊ | 2．0292\％ | 2＊＊1もぐ2＊ | 2－067E2． | 1．76EE， |
| 0021 | $72{ }^{\circ}$ | $C$ | 25 | 20 | 27 | 26 | $2{ }^{\circ}$ | 30 |
| $0021^{\circ}$ | $724 *$ |  | M1．510E2． | 1．27122＊ | 1．10JE2． | 9．43161． | ＊－OSEE1． | 6．802E1． |
| co216 | 125＊ | 6 | 31 | 12 | $3)$ | 3＊ | 35 | 30 |
| $0021{ }^{\circ}$ | $726^{\circ}$ |  | 1 3.875 El ， | S．CIVEI： | 3．647E1． | 2．64＊し， | 1．722E1． | 1．39EI． |
| 00216 | 121＊ | C | 17 | 38 | 39 | ＊ 0 | 41 | 42 |
| 002： | 120＊ |  | 11．02051． | 7．40ヶtz． | 5．4T9E0． | －OYIEO． | 3．001E0． | 2．243E0． |
| 00216 | 1290 | 6 | 4 | ${ }^{*} 4$ | 45 | 46 | 47 | $4{ }^{*}$ |
| 00216 | 13n＊ |  | 11．867と0． | 1•2フプゴ | $9.717 \mathrm{~L}-1$. | 1．4342－1． | S．1／VE－I． | 4．4 14E－1． |
| 00218 | 730 | C |  | ＞0 | ，5： | $32$ | $53$ | $54$ |
| 00216 | 132＊ |  | 13．40」E＝1． | 200＊Ot－1． | 2．0215－1． | 1．5492－1． | 1－1A＊－1． | 9．050E－2， |
| 00216 | 731＊ | 6 | 55 | 3 A | 51 | S＊ | 59 | －C |





| 00230 | - ca* | 6 |
| :---: | :---: | :---: |
| 00236 | -09* |  |
| 00236 | $910^{\circ}$ | 6 |
| 00236 | -110 |  |
| 00234 | -120 | C |
| 00240 | $11^{\circ}$ |  |
| 00240 | 114* | 6 |
| 00240 | $\bigcirc 15{ }^{\circ}$ |  |
| 00240 | -10* | $c$ |
| 00240 | 917 |  |
| 00240 | -1品 | C |
| 00240 | -190 |  |
| 00240 | - $20^{\circ}$ | C |
| 00240 | $9^{\circ} 1^{\circ}$ |  |
| 00840 | -220 | C |
| 00240 | -21* |  |
| 00240 | 92** | C |
| 00240 | -25* |  |
| ~240 | -24* | 6 |
| 00240 | 9210 | c |
| 00240 | -28* | c |
| 00240 | -29* | 6 |
| 00242 | -30* |  |
| 00242 | 9310 | C |
| 00242 | -32* |  |
| 00242 | -31* | $c$ |
| 00242 | 934* |  |
| 00242 | -35* | c |
| 00242 | $136^{\circ}$ |  |
| 00242 | -37* | C |
| 00242 | -3 $8^{\circ}$ |  |
| 00242 | -390 | C |
| 00242 | $40^{\circ}$ |  |
| 00242 | -4 $1^{\circ}$ | C |
| 00242 | $942^{\circ}$ |  |
| $\begin{aligned} & 00242 \\ & 00242 \end{aligned}$ | $\begin{gathered} 9430 \\ 0440 \end{gathered}$ | C |
| 00242 | $75^{5}$ | c |
| 00242 | -46* |  |
| 00242 | -470 | 6 |
| 05242 | 940. |  |
| 00242 | 94. | C |
| 00242 | -50 ${ }^{\circ}$ |  |
| 00242 | -5, ${ }^{\circ}$ | 6 |
| 00242 | 95, |  |
| 09242 | 9530 | C |
| -0フ47 | $954{ }^{\circ}$ |  |
| 00242 | ¢5\% | $c$ |
| 00242 | 9 $5_{6}{ }^{\circ}$ |  |
| 00242 | 75\% | 6 |
| 00242 | - SA. |  |
| 00242 | 9500 | 6 |
| 00242 | -0, |  |
| 00242 | $\bigcirc 0^{\circ}$ | c |
| C0242 | $90^{\circ}$ |  |
| $03^{24}$ | $90^{\circ}$ | 6 |
| 053+2 | 90.4 |  |
| Cosu2 | $905^{\circ}$ | 6 |






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[^0]:    *For sale by the National Technical Information Service, Springfield, Virginia 22161

[^1]:    

