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SPACE SHUTTLE GN&C EQUATION DOCUMENT

No. 25

Conic State Extrapolation

By

Stanley W. Shepperd

William M. Robertson

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**Conic State Extrapolation**

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**Stanley W. Shepperd  
William M. Robertson**

**Charles Stark Draper Laboratory**

**June 1973**

**NAS9-10268**

**for**

**National Aeronautics and Space Administration  
Guidance and Control Systems Branch  
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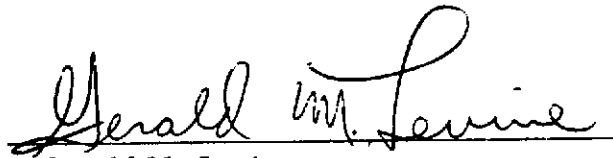
The publication of this report does not constitute approval by the National Aeronautics and Space Administration of the findings or the conclusions contained therein. It is published only for the exchange and stimulation of ideas.

## PREFACE

This document contains the new Unified Conic formulation of R. H. Battin, and is intended to replace the Space Shuttle GN&C Equation Document No. 3 by W. M. Robertson.

## FOREWORD

This document is one of a series of candidates for inclusion in a future revision of JSC -04217, "Space Shuttle Guidance, Navigation and Control Design Equations." The enclosed has been prepared under NAS9-10268, Task No. 15-A, "GN&C Flight Equation Specification Support", and applies to function 1 of the Orbital Coast Navigation Module (ON2) as defined in MSC-03690, Rev. D, "Space Shuttle Orbiter Guidance, Navigation and Control Software Functional Requirements", dated January 1973.

A handwritten signature in cursive script that reads "Gerald M. Levine". The signature is written in black ink and is positioned above a horizontal line.

Gerald M. Levine  
Division Leader, Guidance Analysis  
NASA Programs Department

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## NOMENCLATURE

a	Semi-major axis
A	*Value of $\frac{1}{2} U_2(\tilde{x}; \tilde{\alpha})$ function
b, b <sub>1</sub> , b <sub>2</sub> b <sub>3</sub> , b <sub>5</sub>	Intermediate variables
b <sub>4</sub>	Reciprocal of normalized final position magnitude (b <sub>4</sub> = r <sub>0</sub> /r)
B	*Value of $2 (U_0(\frac{\tilde{x}}{2}; \tilde{\alpha}) + \tilde{\sigma}_0 U_1(\frac{\tilde{x}}{2}; \tilde{\alpha}))$ function
D	*Value of $U_1(\frac{\tilde{x}}{2}; \tilde{\alpha})$ function
e <sub>flag</sub>	Error flag
E	*Value of $U_0(\frac{\tilde{x}}{2}; \tilde{\alpha})$ function
f	True anomaly
f <sub>0</sub>	Coefficient indicating sign of transfer
f <sub>1</sub>	Coefficient to convert to normalized velocity ( $\tilde{\underline{v}} = f_1 \underline{v}$ )
f <sub>2</sub>	Inverse of f <sub>1</sub> ( $\underline{v} = f_2 \tilde{\underline{v}}$ )
f <sub>3</sub>	Coefficient to convert to normalized transfer time interval ( $\Delta\tilde{t} = f_3 \Delta t$ )
f <sub>4</sub>	Inverse of f <sub>3</sub> ( $\Delta t = f_4 \Delta\tilde{t}$ )
f <sub>5</sub>	Coefficient to convert to normalized eccentric anomaly ( $\tilde{x} = f_5 x$ )

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\* These capital letters do not indicate matrices.

$f_6$	Inverse of $f_5$ ( $x = f_6 \tilde{x}$ )
$f_7$	Intermediate variable
$F$	* State extrapolation coefficient
$F_t, \tilde{F}_t$	* State extrapolation coefficients ( $F_t = \sqrt{\frac{\mu}{r_0^3}} \tilde{F}_t$ )
$G, \tilde{G}$	* State extrapolation coefficients ( $G = \sqrt{\frac{r_0^3}{\mu}} \tilde{G}$ )
$G_t$	* State extrapolation coefficient
$i$	Iteration counter
$i_{\max}$	Maximum permissible number of iterations
$\underline{i}_{r_0}$	Unit vector in $\underline{r}_0$ direction
$j$	Counter for calculation of the continued fraction $Q$
$k$	Counter for the summation of the $U_1$ series
$k_{\max}$	Maximum value for $k$
$\tilde{p}$	Normalized semi-latus rectum ( $p = r_0 \tilde{p}$ )
$\tilde{P}$	* Normalized orbital period ( $P = P_{\text{circular}} \tilde{P}$ )
$Q$	* Continued fraction function
$r, \tilde{r}$	Final position magnitude and its normalized analog ( $r = r_0 \tilde{r}$ )
$\underline{r}(t)$	Inertial position vector corresponding to time $t$
$r_0$	Magnitude of $\underline{r}_0$

---

\* These capital letters do not indicate matrices.



$\underline{r}_0$	Inertial position vector corresponding to time $t_0$
$s_{\max}$	Maximum scalar representable on the computer
$s_{\min}$	Minimum positive scalar representable on the computer
$u_1$	Value of $U_1(\frac{\tilde{x}}{4}; \tilde{\alpha})$ function
$u_{1old}$	Previous value of $u_1$
$U_n(x; \alpha)$	* Universal function of Battin
$\underline{v}(t)$	Inertial velocity vector corresponding to time $t$
$\underline{v}_0, \tilde{v}_0$	Inertial velocity vector, corresponding to initial time $t_0$ , and its normalized analog ( $\underline{v}_0 = v_{\text{circular}} \tilde{v}_0$ )
$w$	Value of $U_0(\frac{\tilde{x}}{4}; \tilde{\alpha})$ function
$x, \tilde{x}$	Eccentric anomaly difference and its normalized analog ( $x = \sqrt{r_0} \tilde{x}$ , independent variable used in the Kepler iteration scheme)
$\tilde{x}_{\text{arg}}$	Independent variable used as the input argument to the Transfer Time Interval Routine
$x_c$	Value of $x$ to which the Kepler iteration scheme converged
$x'_c$	Previous value of $x_c$
$\tilde{x}_{\text{guess}}$	Guess of $\tilde{x}$
$\tilde{x}_{\text{last}}$	Previous value of $\tilde{x}$

---

\* This capital letter does not indicate a matrix.

$x_{\max}, \tilde{x}_{\max}$	Independent variable upper bound, and its normalized analog
$x_{\min}, \tilde{x}_{\min}$	Independent variable lower bound, and its normalized analog
$\tilde{x}_{\text{old}}$	Value of $\tilde{x}_{\text{guess}}$ from the previous iteration
$\tilde{x}_p$	Value of $\tilde{x}$ corresponding to one period of the orbit
$x_q, y$	Intermediate variables
$z, \tilde{z}$	Values of $U_1(\frac{x}{4}; \alpha), U_1(\frac{\tilde{x}}{4}; \tilde{\alpha})$ functions ( $z = \sqrt{r_0} \tilde{z}$ )
$\alpha, \tilde{\alpha}$	Reciprocal of the semi-major axis and its normalized analog ( $\tilde{\alpha} = r_0 \alpha$ )
$\Delta t, \tilde{\Delta t}$	Transfer time interval and its normalized analog ( $\Delta t = P_{\text{circular}} \tilde{\Delta t}$ )
$\tilde{\Delta t}_{\text{arg}}$	Dependent variable used as the output argument from the Transfer Time Interval Routine
$\Delta t_c$	Value of the transfer time interval calculated in the Kepler Routine
$\Delta t'_c$	Previous value of $\Delta t_c$
$\tilde{\Delta t}_{\text{error}}$	Error between $\tilde{\Delta t}_{\text{guess}}$ and $\tilde{\Delta t}$ during any particular iteration
$\tilde{\Delta t}_{\text{guess}}$	Transfer time interval corresponding to the most recent independent variable $\tilde{x}_{\text{guess}}$ during the Kepler iteration
$\tilde{\Delta t}_{\text{last}}$	Previous value of $\tilde{\Delta t}$
$\Delta t'_{\text{max}}$	Dependent variable boundary distance on the maximum side used in the Secant Iterator
$\Delta t'_{\text{min}}$	Dependent variable boundary distance on the minimum side used in the Secant Iterator

$\Delta t_{\max}$ , $\tilde{\Delta t}_{\max}$	Dependent variable upper bound and its normalized analog
$\Delta t_{\min}$ , $\tilde{\Delta t}_{\min}$	Dependent variable lower bound and its normalized analog
$\tilde{\Delta t}_{\text{old}}$	Value of $\tilde{\Delta t}_{\text{guess}}$ from the previous iteration
$\Delta u_1$	Increment in $u_1$
$\tilde{\Delta x}$	Increment in $\tilde{x}_{\text{guess}}$
$\epsilon_\alpha$	Tolerance defining near parabolic orbits used in boundary selection
$\tilde{\epsilon}_t$	Primary convergence criterion: error in normalized transfer time interval
$\epsilon'_t$	Secondary convergence criterion: minimum permissible difference between two transfer time intervals used in the Secant Iterator
$\epsilon_w$	Tolerance defining area where the computer cannot handle the continued fraction $Q$ calculation
$\theta$	Transfer angle (true anomaly increment)
$\mu$	Gravitational parameter of the earth
$\sigma_0, \tilde{\sigma}_0$	Conic parameter and its normalized analog ( $\sigma_0 = \sqrt{r_0} \tilde{\sigma}_0$ )

1.        INTRODUCTION

The Conic State Extrapolation Routine provides the capability to conically extrapolate any spacecraft inertial state vector either backwards or forwards as a function of time or as a function of transfer angle. It is merely the coded form of two versions of the solution of the two-body differential equations of motion of the spacecraft center of mass. Because of its relatively fast computation speed and moderate accuracy, it serves as a preliminary navigation tool and as a method of obtaining quick solutions for targeting and guidance functions. More accurate (but slower) results are provided by the Precision State Extrapolation Routine.

## 2. FUNCTIONAL FLOW DIAGRAMS

The Conic State Extrapolation Routine basically consists of two parts - one for extrapolating in transfer time interval, and one for extrapolating in transfer angle. Some portions of the formulation are common to the two parts and may be arranged as subroutines on a computer. Similarly, there are several areas of overlap with the Conic Required Velocity Routine (Ref. 5), and these may be treated likewise.

### 2.1 Conic State Extrapolation as a Function of Transfer Time Interval - Kepler Routine

This routine involves a single loop iterative procedure, and hence is organized into three sections - initialization, iteration, and final computations. This scheme is shown in Figure 1, where the variable " $\tilde{x}_{\text{guess}}$ " is the independent variable used in the iteration procedure. For a given initial state, the variable " $\tilde{x}_{\text{guess}}$ " measures the amount of transfer along the extrapolated trajectory. Hence, the iteration involves adjusting " $\tilde{x}_{\text{guess}}$ " until the transfer time interval calculated from it agrees, to within some tolerance, with the specified transfer time interval. Then the extrapolated state vector is calculated from several intermediate variables already calculated in the transfer time interval computation.

### 2.2 Conic State Extrapolation as a Function of Transfer Angle - Theta Routine

This routine involves a direct calculation (i. e. , does not have an iteration scheme), as shown in Figure 2. The extrapolated state vector is again calculated from the same intermediate variables used in the Kepler Routine. However, this time these variables are computed directly in terms of the initial state and the transfer angle.

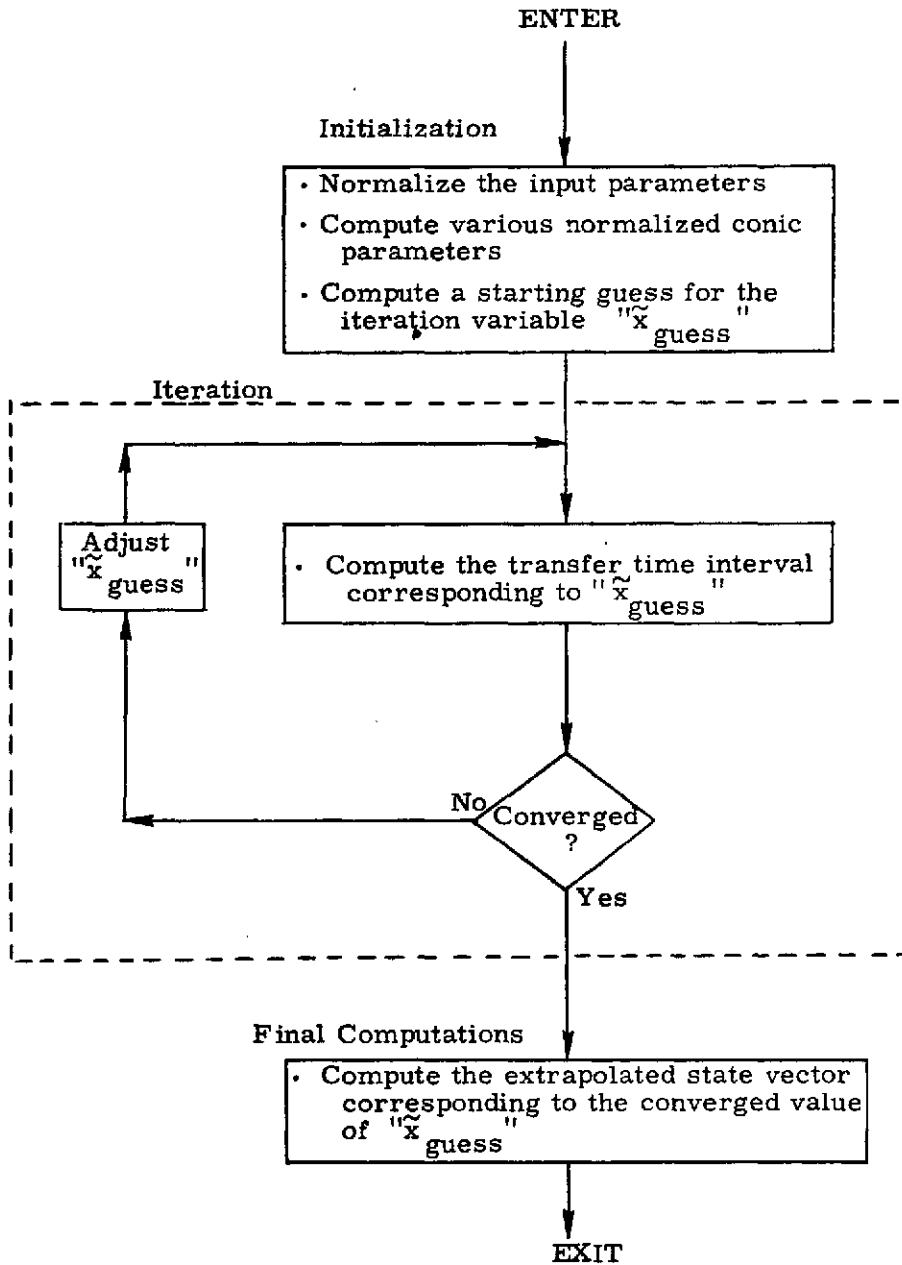


Figure 1. Kepler Routine, Functional Flow Diagram

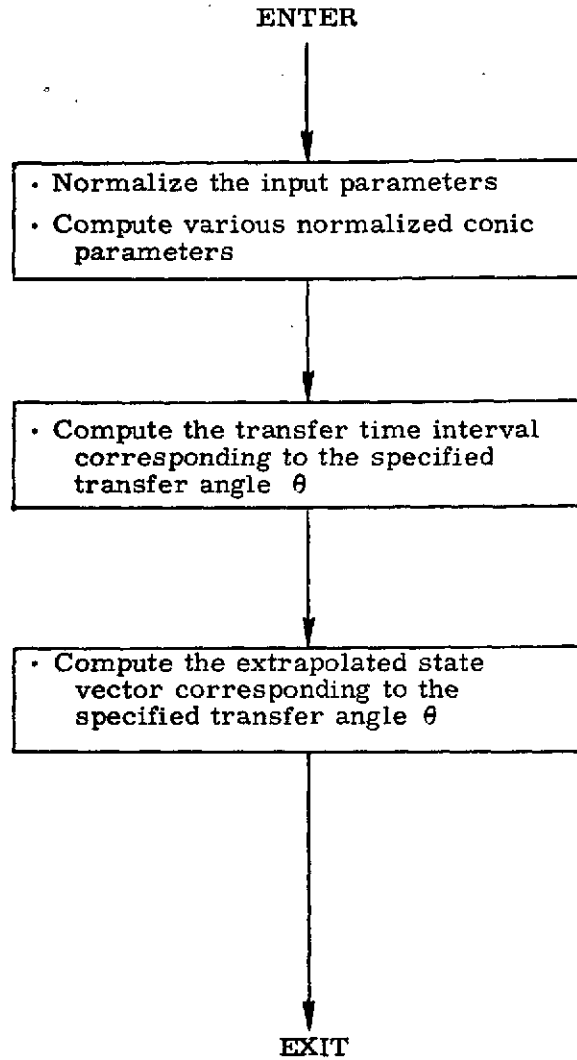


Figure 2. Theta Routine, Functional Flow Diagram

### 3. INPUT AND OUTPUT VARIABLES

The Conic State Extrapolation Routine has only one universal constant: the gravitational parameter of the earth. Its principal input variables are the inertial state vector which is to be extrapolated and the transfer time interval or transfer angle through which the extrapolation is to be made. Several optional input variables may be supplied in the transfer time case in order to speed the computations. The principal output variable of both cases is the extrapolated inertial state vector.

#### 3.1 Conic State Extrapolation as a Function of Transfer Time Interval - Kepler Routine

##### Input Variables

- $(\underline{r}_0, \underline{v}_0)$  Inertial state vector which is to be extrapolated (corresponds to time  $t_0$ ).
- $\Delta t$  Transfer time interval through which the extrapolation is to be made.
- $x$  Starting guess of the independent variable (used to speed convergence). If no guess is available, set  $x = 0$ , and the routine will generate its own guess.
- $\Delta t'_c$  Value of dependent variable, the transfer time interval, to which the previous call to Kepler had converged.
- $x'_c$  Value of the independent variable, universal eccentric anomaly difference, to which the previous call to Kepler had converged.

##### Output Variables

- $(\underline{r}, \underline{v})$  Extrapolated inertial state vector (corresponds to time  $t$ ).
- $\Delta t_c$  Converged value of the dependent variable, the transfer time interval, from the Kepler iteration scheme (should agree closely with  $\Delta t$ ).



$x_c$  Converged value of the independent variable from the Kepler iteration scheme.

$e_{flag}$  Error flag =  $\left\{ \begin{array}{l} 0 \text{ no error} \\ 1 \text{ solution failed to converge} \\ 2 \text{ } U_1 \text{ series failed to converge} \\ 3 \text{ both 1 and 2 occurred} \end{array} \right.$

### 3.2 Conic State Extrapolation as a Function of Transfer Angle - Theta Routine

$(\underline{r}_0, \underline{v}_0)$  Inertial state vector which is to be extrapolated.

$\theta$  Transfer angle through which the extrapolation is to be made.

$(\underline{r}, \underline{v})$  Extrapolated inertial state vector .

$\Delta t_c$  Transfer Time Interval corresponding to the conic extrapolation through the transfer angle  $\theta$ .

$e_{flag}$  Error flag =  $\left\{ \begin{array}{l} 0 \text{ no error} \\ 1 \text{ orbit too nearly rectilinear} \\ 2 \text{ multi-revolution input for hyperbolic trajectory} \\ 3 \text{ transfer past asymptote on a hyperbolic trajectory} \\ 4 \text{ transfer required closure through infinity} \end{array} \right.$

#### 4. DESCRIPTION OF EQUATIONS

The universal formulation of Battin in terms of the universal eccentric anomaly difference is used. It should be noted that the formulation given in the detailed flow diagrams of Section 5 is a normalized version of the formulation presented here. It is felt that the equation descriptions of this section are more clear when expressed in standard variables rather than in normalized variables.

The universal eccentric anomaly, usually denoted by  $x$ , is defined by the relations

$$x = \begin{cases} \sqrt{a} (E - E_0) & \text{ellipse} \\ \sqrt{p} (\tan f/2 - \tan f_0/2) & \text{parabola} \\ \sqrt{-a} (H - H_0) & \text{hyperbola} \end{cases}$$

Here,  $a$  is the semi-major axis,  $E$  and  $H$  are the eccentric anomaly and its hyperbolic analog,  $p$  is the semi-latus rectum and  $f$  the true anomaly. Using this definition, the generalized form of Kepler's equation expresses the transfer time interval as

$$\Delta t = \frac{1}{\sqrt{\mu}} [r_0 U_1(x; \alpha) + \sigma_0 U_2(x; \alpha) + U_3(x; \alpha)]$$

where 
$$U_n(x; \alpha) = x^n \left( \frac{1}{n!} - \frac{\alpha x^2}{(n+2)!} + \frac{(\alpha x^2)^2}{(n+4)!} - \dots \right)$$

and 
$$\alpha = \frac{1}{a} = \frac{2}{r_0} - \frac{v_0^2}{\mu}$$

and 
$$\sigma_0 = \frac{r_0 \cdot v_0}{\sqrt{\mu}}$$

Furthermore, the expressions for the extrapolated position and velocity vectors ( $\underline{r}$ ,  $\underline{v}$ ) in terms of the initial position and velocity vectors ( $\underline{r}_0$ ,  $\underline{v}_0$ ) are also given in terms of  $x$  by the following standard formulae

$$\begin{aligned} r &= r_0 U_0(x; \alpha) + \sigma_0 U_1(x; \alpha) + U_2(x; \alpha) \\ \underline{r}(t) &= \left[ 1 - \frac{1}{r_0} U_2(x; \alpha) \right] \underline{r}_0 + \left[ \frac{r_0 U_1(x; \alpha) + \sigma_0 U_2(x; \alpha)}{\sqrt{\mu}} \right] \underline{v}_0 \\ \underline{v}(t) &= \left[ -\frac{\sqrt{\mu}}{r r_0} U_1(x; \alpha) \right] \underline{r}_0 + \left[ 1 - \frac{1}{r} U_2(x; \alpha) \right] \underline{v}_0 \end{aligned}$$

However, in this document a slightly modified, but equivalent, formulation is used, namely

$$\Delta t = \frac{U_1(\frac{x}{2}; \alpha)}{\sqrt{\mu}} [2(r_0 U_0(\frac{x}{2}; \alpha) + \sigma_0 U_1(\frac{x}{2}; \alpha)) + \frac{1}{2} U_2(x; \alpha) Q]$$

$$r = r_0 + (1 - \alpha r_0) U_2(x; \alpha) + 2 \sigma_0 U_0(\frac{x}{2}; \alpha) U_1(\frac{x}{2}; \alpha)$$

$$\underline{r}(t) = F \underline{r}_0 + G \underline{v}_0$$

$$\underline{v}(t) = F_t \underline{r}_0 + G_t \underline{v}_0$$

Here, the function Q is defined in Section 4.1, but the other variables are as follows:

$$F = 1 - \frac{1}{r_0} U_2(x; \alpha)$$

$$G = \frac{2 r_0 U_0(\frac{x}{2}; \alpha) U_1(\frac{x}{2}; \alpha) + \tilde{\sigma}_0 U_2(x; \alpha)}{\sqrt{\mu}}$$

$$F_t = -\frac{2 \sqrt{\mu}}{r r_0} U_0(\frac{x}{2}; \alpha) U_1(\frac{x}{2}; \alpha)$$

$$G_t = 1 - \frac{1}{r} U_2(x; \alpha)$$

The family of functions  $U_n(x; \alpha)$  used in these equations can be related to some elementary functions by noting that

$$U_0(x; \alpha) = \begin{cases} \cos \sqrt{\alpha} \cdot x & \alpha > 0 \\ \cosh \sqrt{-\alpha} x & \alpha < 0 \end{cases}$$

$$U_1(x; \alpha) = \begin{cases} \frac{\sin \sqrt{\alpha} \cdot x}{\sqrt{\alpha}} & \alpha > 0 \\ \frac{\sinh \sqrt{-\alpha} x}{\sqrt{-\alpha}} & \alpha < 0 \end{cases}$$

Then, from these the higher order functions follow directly from the useful identity

$$U_n(x; \alpha) + \alpha U_{n+2}(x; \alpha) = \frac{x^n}{n!}$$

#### 4.1 The Q Function

A very useful function employed in the evaluation of the universal formulation of Battin is the Q function which may be written

$$Q = \frac{U_3(x; \alpha)}{U_1\left(\frac{x}{2}; \alpha\right)}$$

It can be shown that Q is a hypergeometric function of the variable

$$q = \alpha U_1^2\left(\frac{x}{4}; \alpha\right)$$

From this, Battin has developed a continued fraction formulation for evaluating Q as a function of the variable

$$w = U_0\left(\frac{x}{4}; \alpha\right)$$

to improve the convergence. The formulation is conveniently expressed by

$$Q = \frac{1}{w} \left[ 1 + \frac{2}{3} \frac{1}{w(w+1)} \left( 1 - \frac{B_1}{4} \right) \right]$$

where

$$B_{n-1} = \frac{\frac{w-1}{w+1}}{1 + \left(\frac{n-1}{n+2}\right) (1 - B_n)}$$

Evaluation of this expression is started by setting  $B_M = 0$  for a sufficiently large value of M, and calculating lower  $B_n$ 's recursively until  $B_1$  is reached. This means that the number of levels M of the continued fraction necessary to achieve some specified accuracy must be precomputed as a function of w.

#### 4.2 Conic State Extrapolation as a Function of Transfer Time Interval - Kepler Routine

Since the transfer time interval  $\Delta t$  is given, it is desired to find the x corresponding to it in the generalized Kepler equation. Then the extrapolated state vector  $(\underline{r}, \underline{v})$  expression can be evaluated based on that value of x. Unfortunately, the generalized Kepler equation expresses  $\Delta t$  as a transcendental function of x, and no power series inversion of the equation is known which has good convergence properties for all orbits. Therefore, it is necessary to solve the equation iteratively for x.

The actual evaluation of the Kepler equation is conveniently expressed in terms of the variable

$$z = 2 U_1 \left( \frac{x}{2}; \alpha \right)$$

From this it can be shown that

$$U_0 \left( \frac{x}{4}; \alpha \right) = \sqrt{1 - \frac{\alpha}{4} z^2}$$

$$U_0 \left( \frac{x}{2}; \alpha \right) = 1 - \frac{\alpha}{2} z^2$$

$$U_1 \left( \frac{x}{2}; \alpha \right) = w z$$

$$U_2 (x; \alpha) = 2 w^2 z^2$$

These quantities are needed to evaluate the transfer time interval corresponding to  $x$  during each iteration. Then, once convergence is achieved the latest values of these quantities are used to calculate the extrapolated state vectors.

The iteration scheme used to solve for  $x$  is a modified secant method (linear inverse interpolation/extrapolation). It merely finds a new guess for the independent variable  $x_{\text{new}}$  which attempts to adjust the dependent variable  $\Delta t$  to the desired value. This adjustment is based on a linear interpolation/extrapolation of the last guess of the independent variable  $x_{\text{old}}$ . The scheme requires a minimum point and a maximum point on the  $\Delta t$  vs.  $x$  curve between which the solution is known to be. These two points define two possible guesses for the new  $x$ , namely

$$x_{\text{new}} = x_{\text{old}} + \left( \frac{\Delta t - \Delta t_{\text{old}}}{\Delta t_{\text{old}} - \Delta t_{\text{min}}} \right) (x_{\text{old}} - x_{\text{min}})$$

$$x_{\text{new}} = x_{\text{old}} + \left( \frac{\Delta t - \Delta t_{\text{old}}}{\Delta t_{\text{old}} - \Delta t_{\text{max}}} \right) (x_{\text{old}} - x_{\text{max}})$$

one of which is an interpolation and the other an extrapolation. The modified scheme preferentially selects the extrapolated guess from the two possibilities. However, if this happens to extrapolate outside the upper or lower bounds, then the interpolated guess is used to ensure a solution. It should be noted that these bounds are continually reset during the iteration as more and more values of  $x$  are determined to be too small or too large. This modification has been found to improve the convergence when the desired transfer time interval is near a "knee" in the  $\Delta t$  vs.  $x$  curve.

### 4.3 Conic State Extrapolation as a Function of Transfer Angle - Theta Routine

As with the Kepler Routine, the universal formulation of Battin in terms of the universal eccentric anomaly difference  $x$  is used in the Theta Routine. However, in this case the functions  $U_n(x; \alpha)$  may be expressed directly in terms of the specified transfer angle. The relationships used in this formulation are

$$U_0\left(\frac{x}{2}; \alpha\right) = \sqrt{\frac{r}{r_0}} \left[ \cos \frac{\theta}{2} - \frac{\sigma_0}{\sqrt{p}} \sin \frac{\theta}{2} \right]$$

$$U_1\left(\frac{x}{2}; \alpha\right) = \sqrt{\frac{r r_0}{p}} \sin \frac{\theta}{2}$$

$$U_2(x; \alpha) = \frac{2 r r_0}{p} \sin^2 \frac{\theta}{2}$$

However, to evaluate these functions requires the final position magnitude, but from the position magnitude extrapolation equation it can be shown that

$$\frac{r_0}{r} = 2 \left[ \cos \frac{\theta}{2} \left( \cos \frac{\theta}{2} - \frac{\sigma_0}{\sqrt{p}} \sin \frac{\theta}{2} \right) + \frac{r_0}{p} \sin^2 \frac{\theta}{2} \right] - 1$$

Thus, from these quantities the extrapolated state vectors are calculated using the formulae in Section 4. Furthermore, these quantities may also be used to evaluate the transfer time interval from the generalized Kepler equation.

## 5. DETAILED FLOW DIAGRAMS

This section contains detailed flow diagrams of two Conic State Extrapolation Routines (Kepler and Theta) and the subroutines used by them.

Each input and output variable in the routine and subroutine call statements can be followed by a symbol in brackets. This symbol identifies the notation for the corresponding variable in the detailed description and flow diagrams of the called routine. When identical notation is used, the bracket symbol is omitted.

The Kepler routine utilizes the following subroutines, which are diagrammed in Section 5.3:

- Kepler Transfer Time Interval (Section 5.3.1)
  - $U_1$  Series Summation (Section 5.3.2)
  - $Q$  Continued Fraction (Section 5.3.3)
- Kepler Iteration Loop (Section 5.3.4)
  - Secant Iterator (Section 5.3.5)
  - Kepler Transfer Time Interval (Section 5.3.1)
    - $U_1$  Series Summation (Section 5.3.2)
    - $Q$  Continued Fraction (Section 5.3.3)
- Extrapolated State Vector (Section 5.3.6)

The Theta routine utilizes the following subroutines, which are diagrammed in Section 5.3:

- $Q$  Continued Fraction (Section 5.3.3)
- Extrapolated State Vector (Section 5.3.6)

5.1 Conic State Extrapolation as a Function of Transfer Time - Kepler Routine

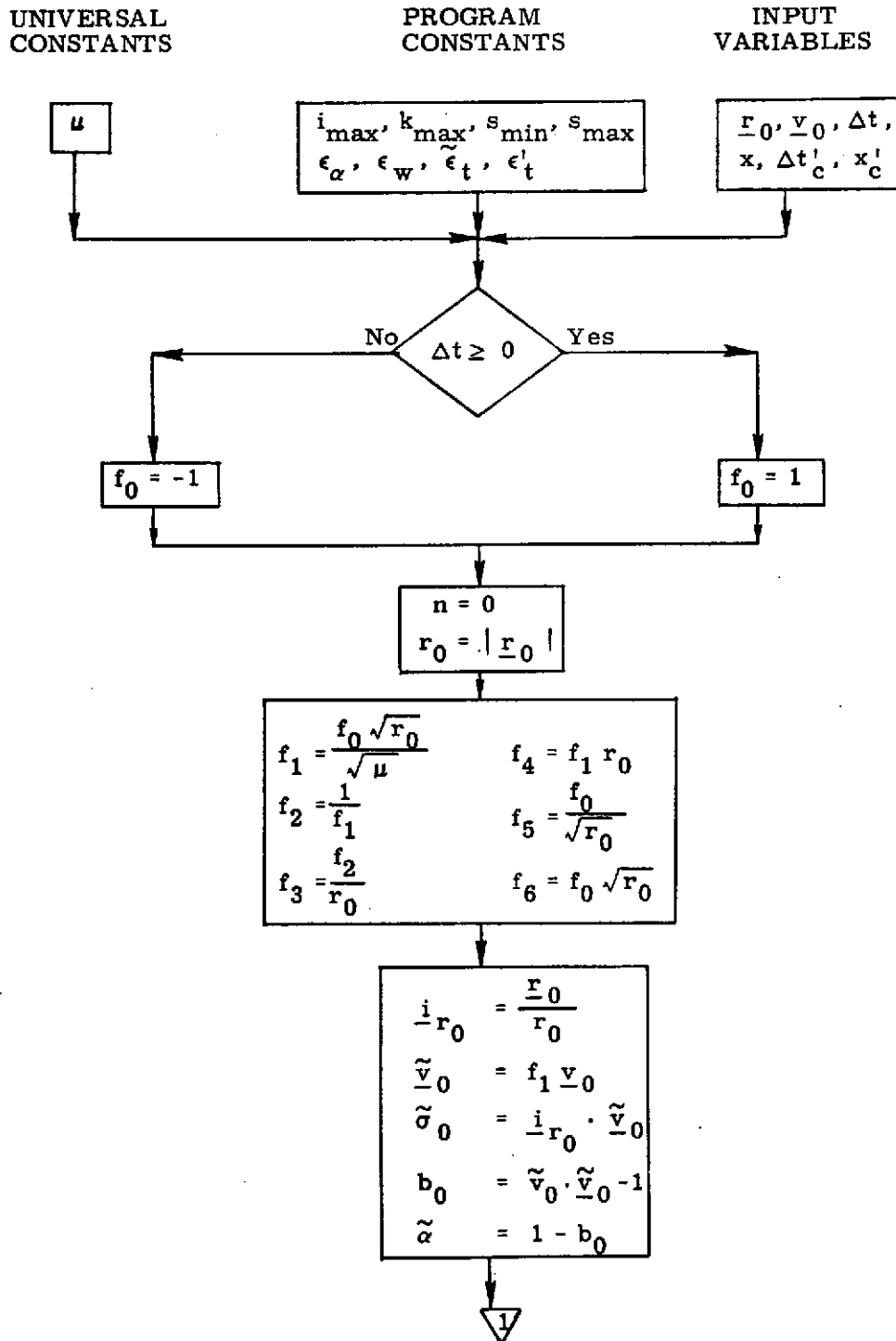


Figure 3a. Kepler Routine, Detailed Flow Diagram



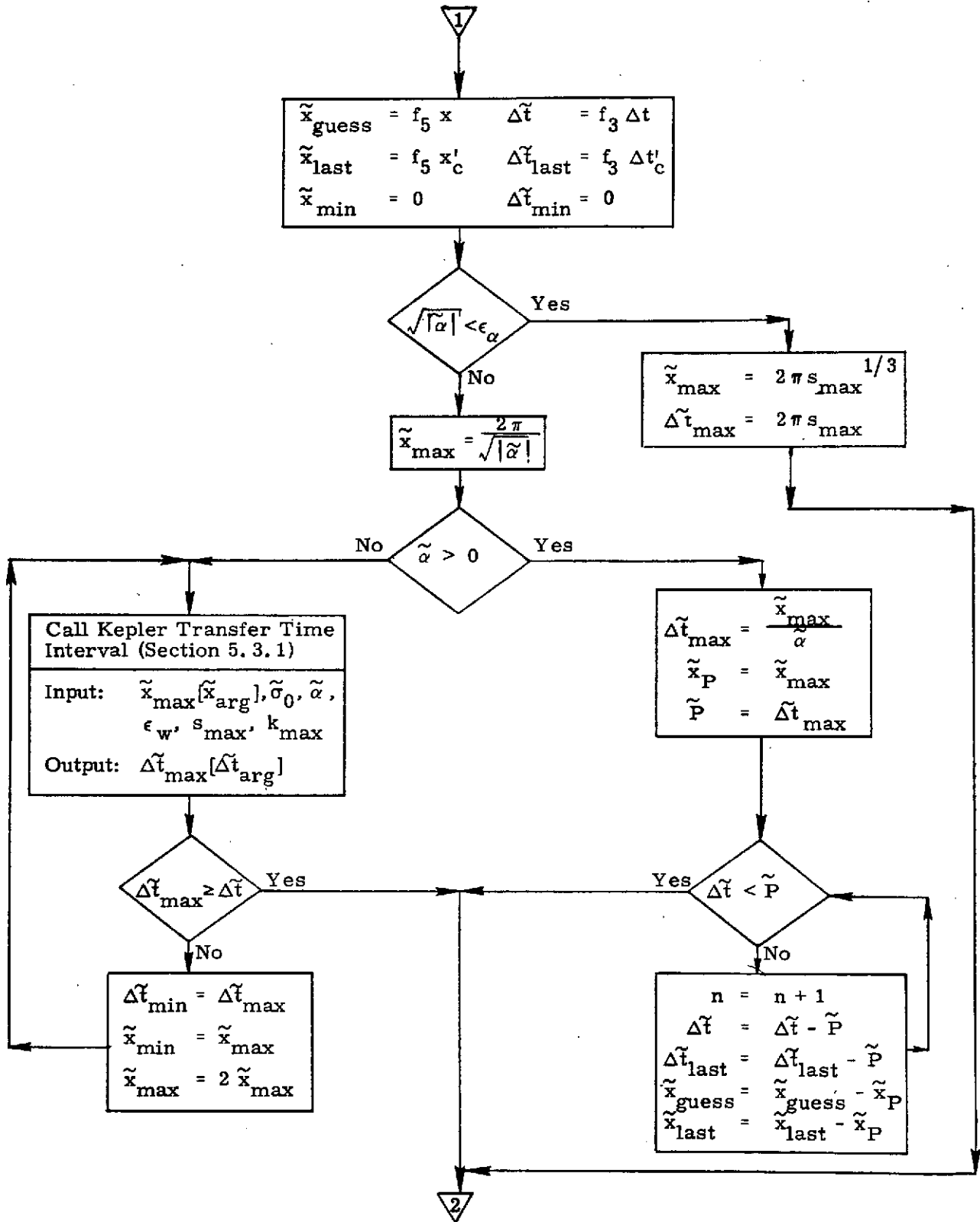


Figure 3b. Kepler Routine, Detailed Flow Diagram

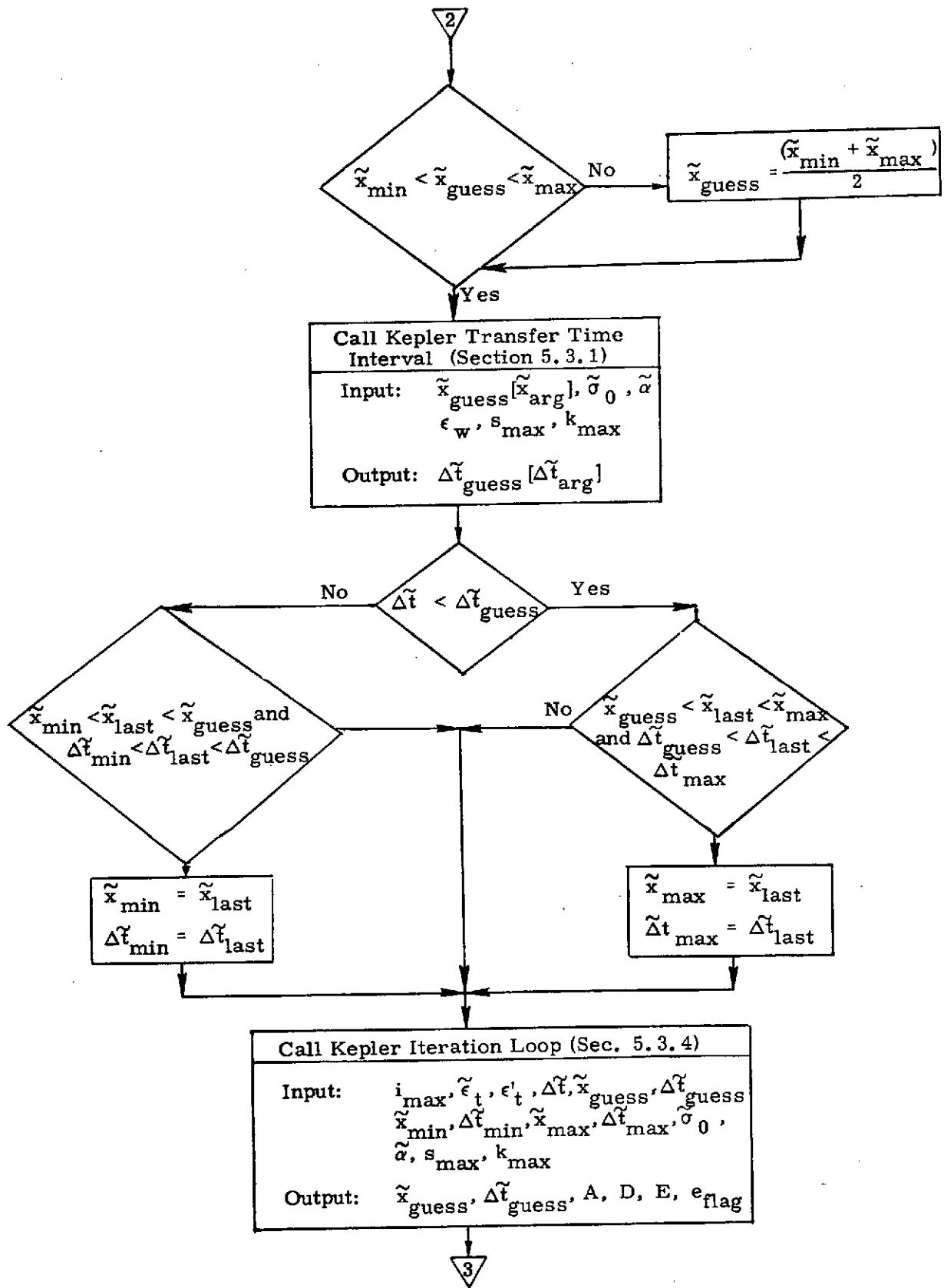


Figure 3c. Kepler Routine, Detailed Flow Diagram

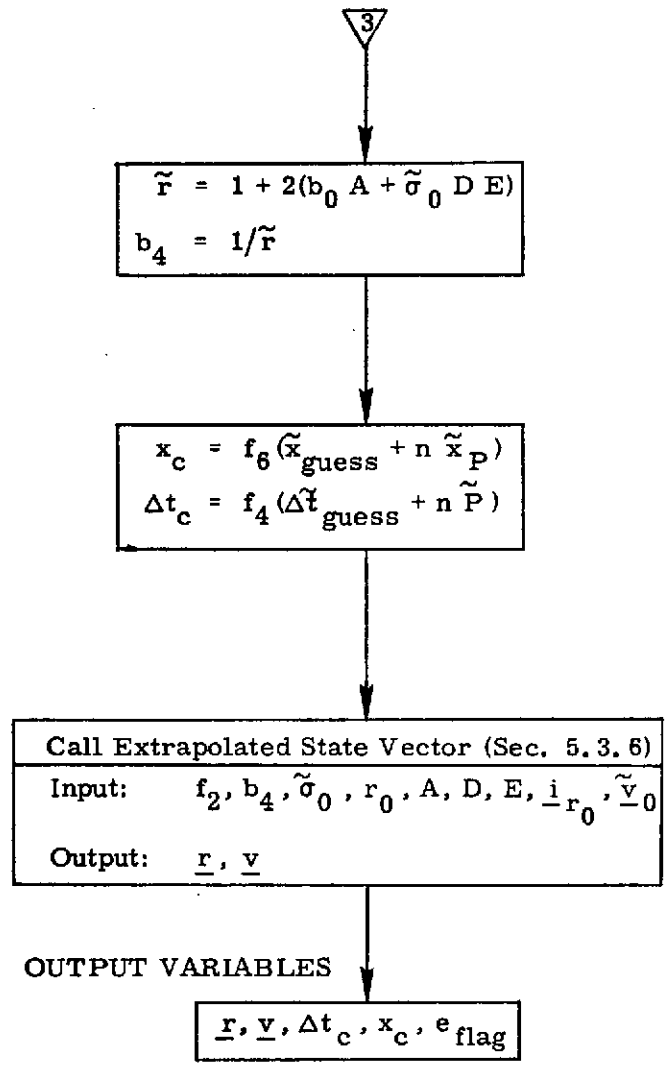


Figure 3d. Kepler Routine, Detailed Flow Diagram

5.2 Conic State Extrapolation as a Function of Transfer Angle - Theta Routine

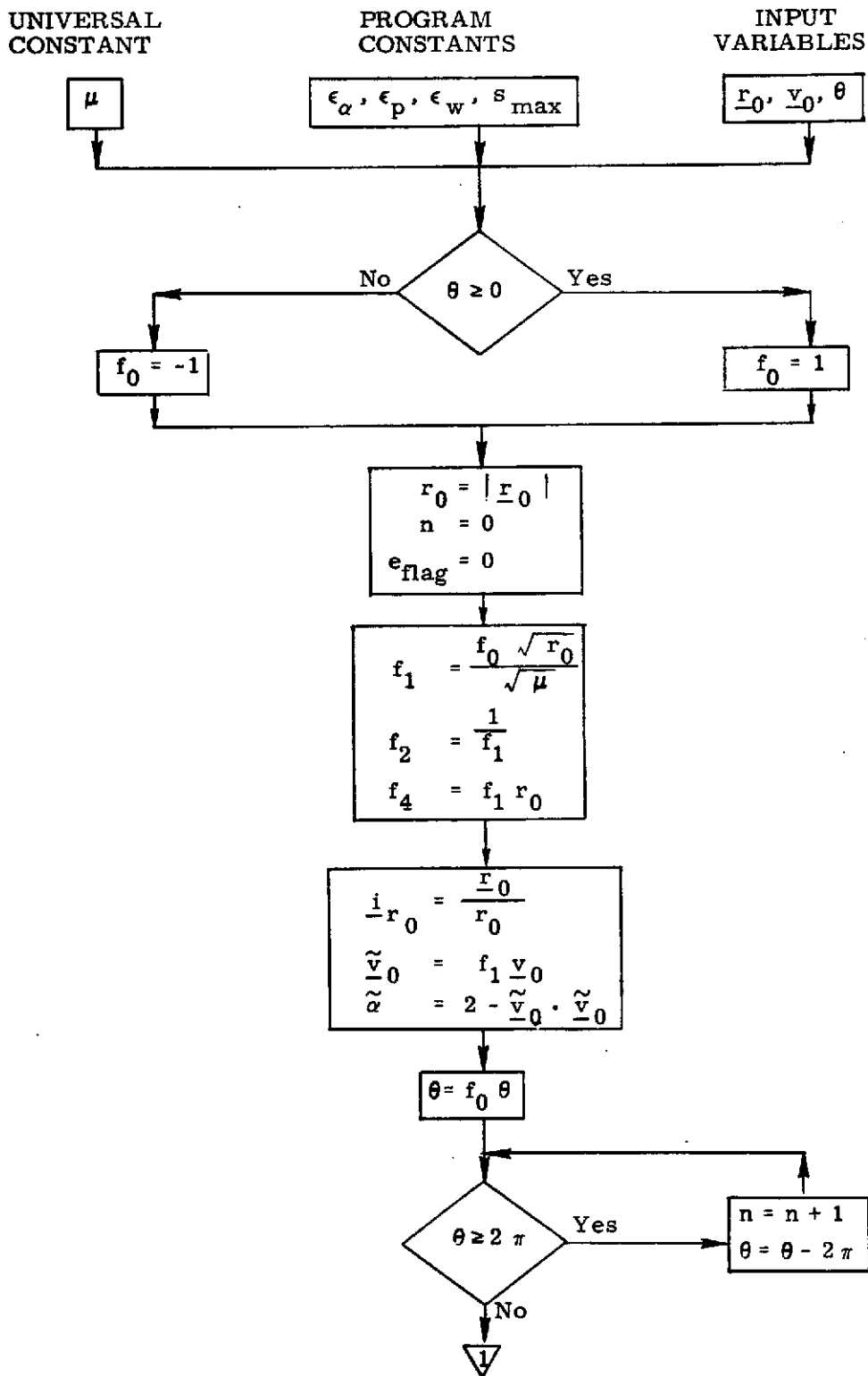


Figure 4a. Theta Routine, Detailed Flow Diagram

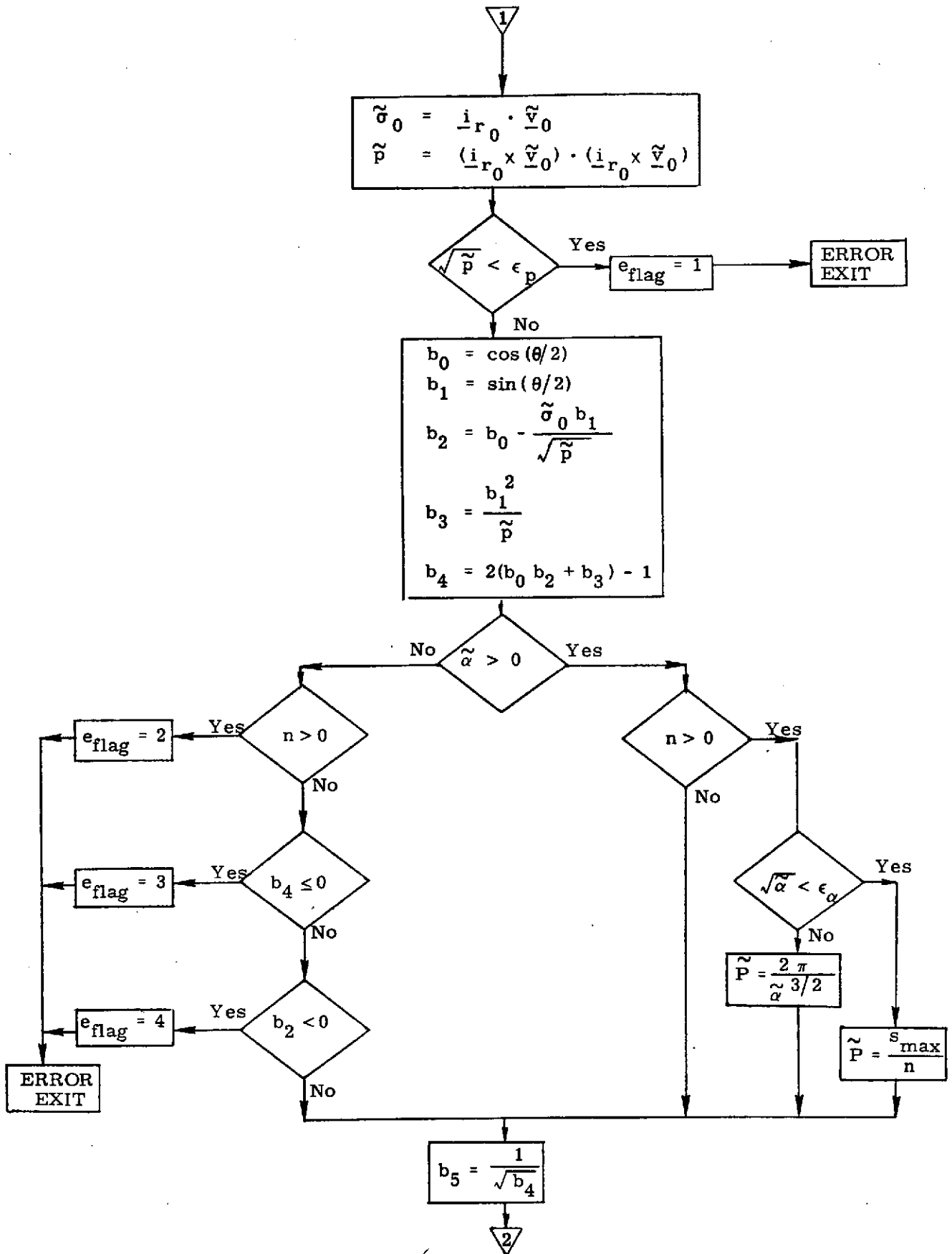


Figure 4b. Theta Routine, Detailed Flow Diagram

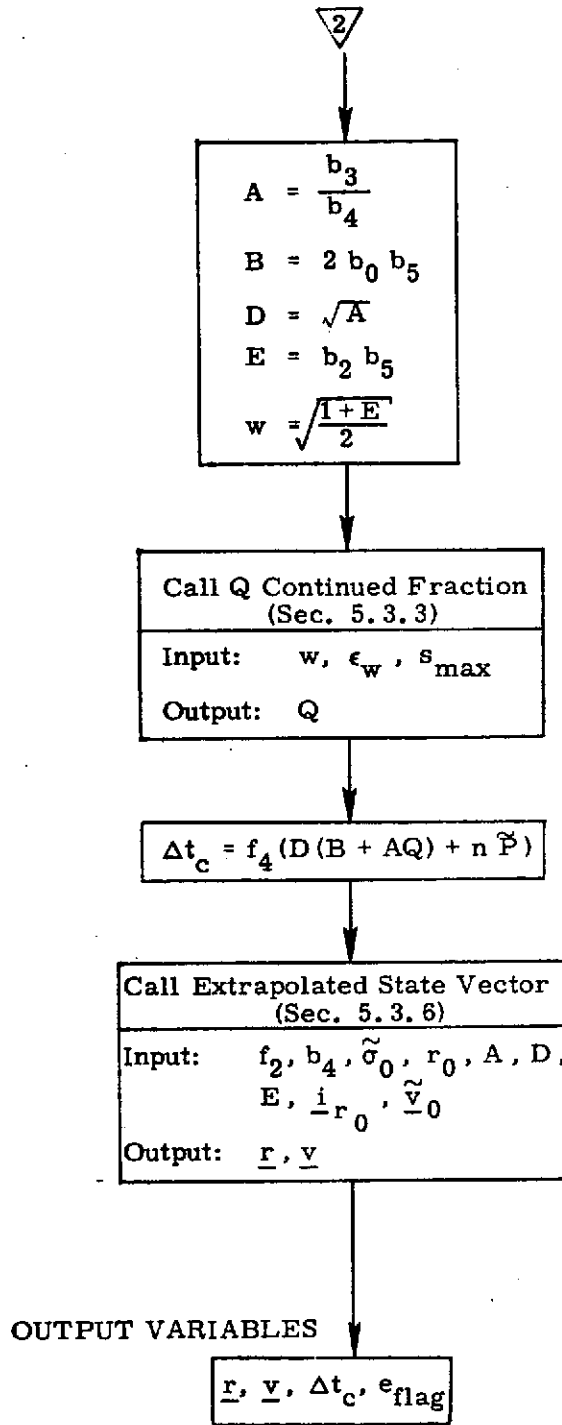


Figure 4c. Theta Routine, Detailed Flow Diagram

5.3 Subroutines Used by the Transfer Time and/or the Transfer Angle Conic Extrapolation Routines

5.3.1 Kepler Transfer Time Interval Subroutine

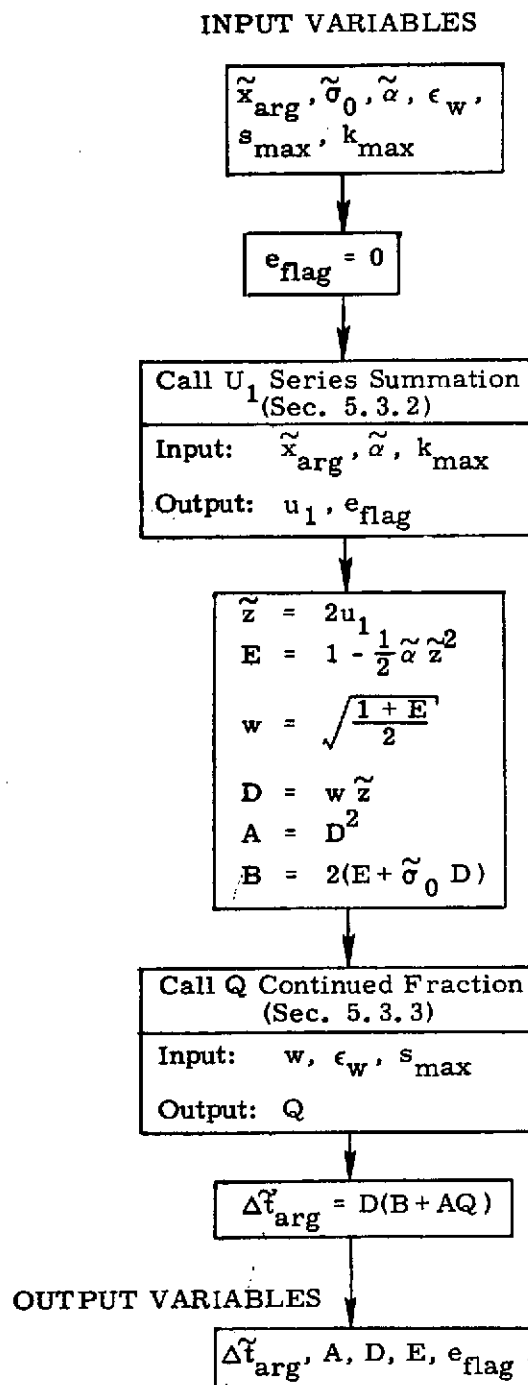


Figure 5. Kepler Transfer Time Interval, Detailed Flow Diagram

5.3.2 U<sub>1</sub> Series Summation Subroutine

INPUT VARIABLES

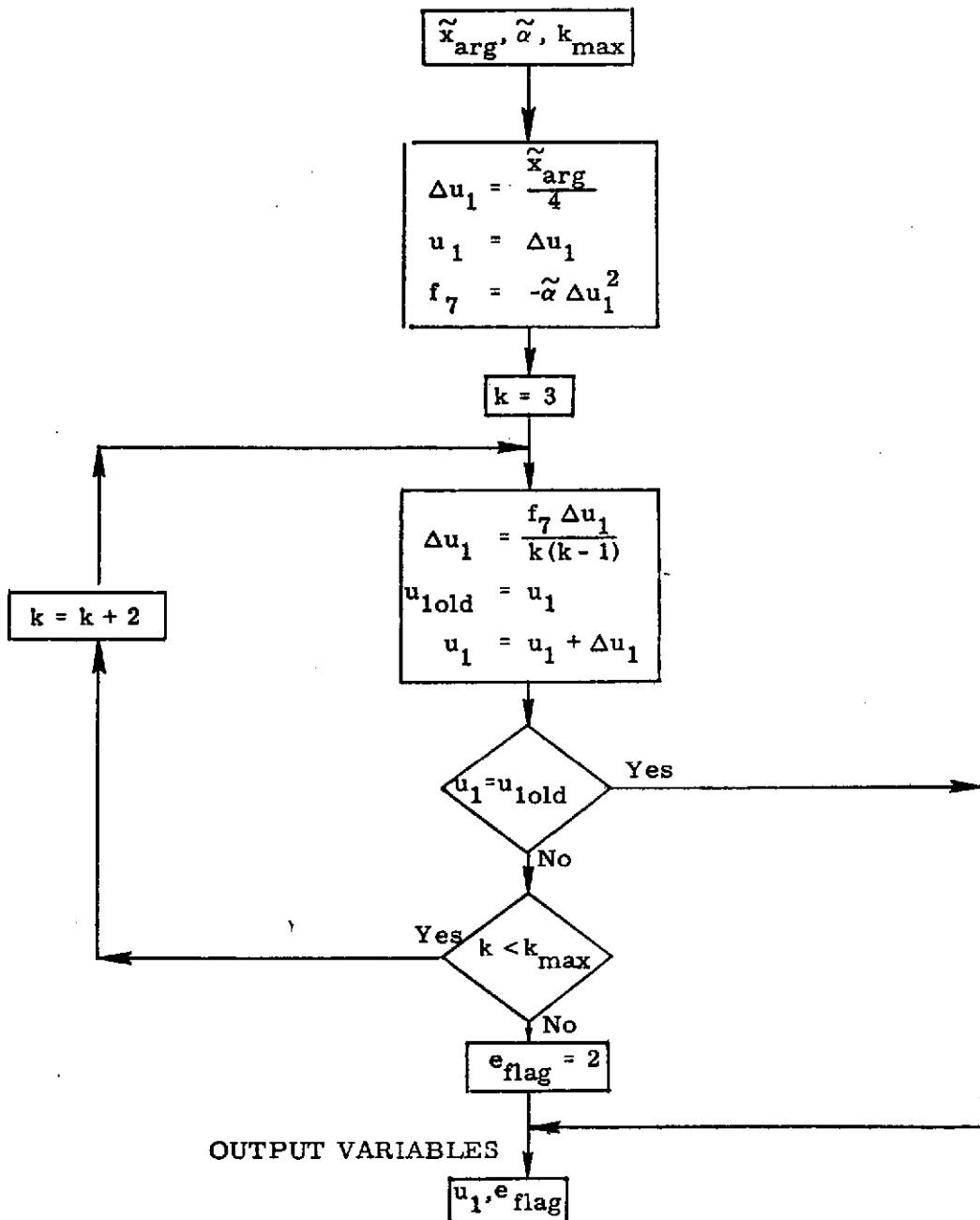


Figure 6. U<sub>1</sub> Series Summation, Detailed Flow Diagram



### 5.3.3 Q Continued Fraction Subroutine

This subroutine is identical to the one used in the Conic Required Velocity routines. It should be noted that this routine has been designed to give double precision (16 digits) accuracy.

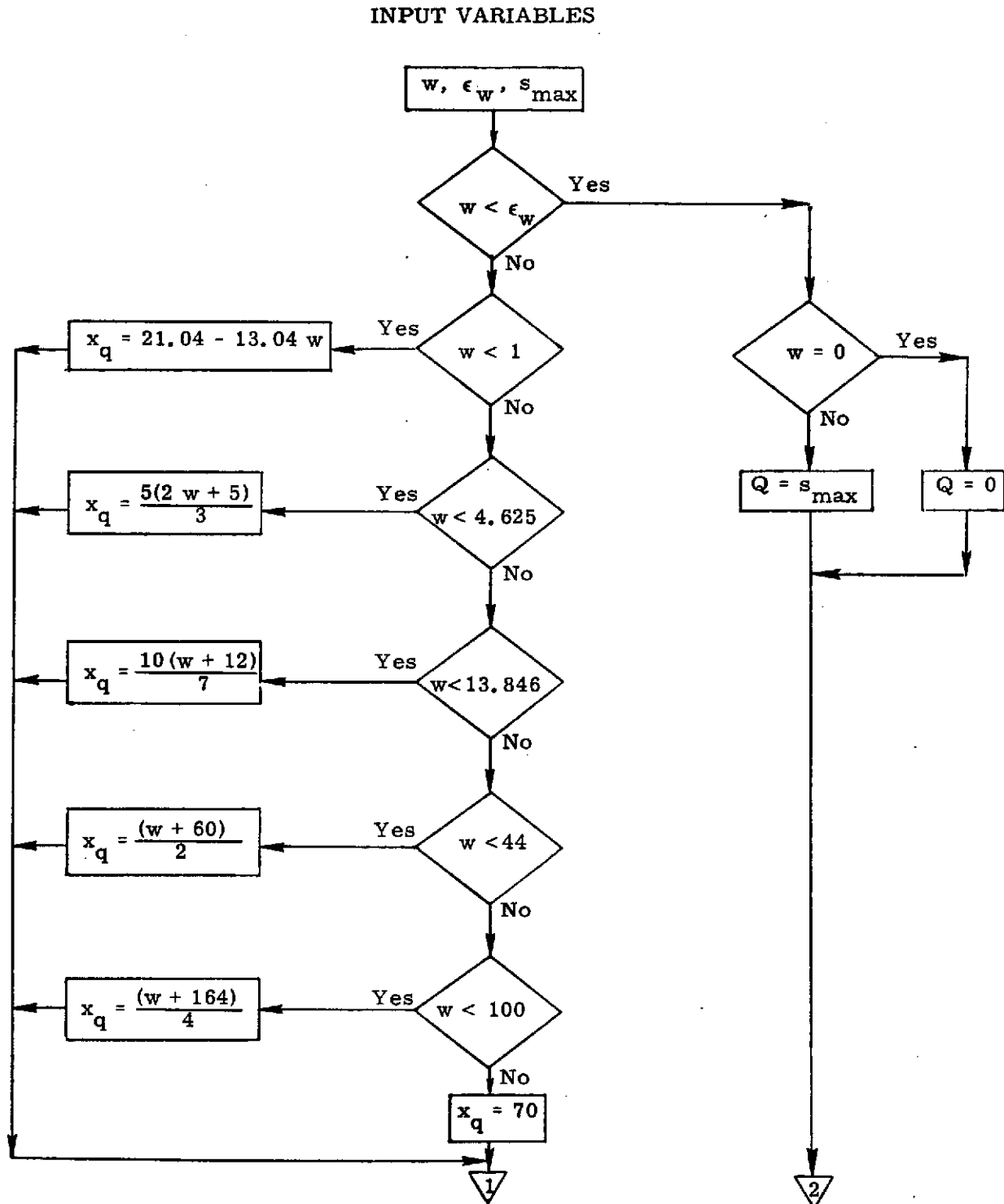
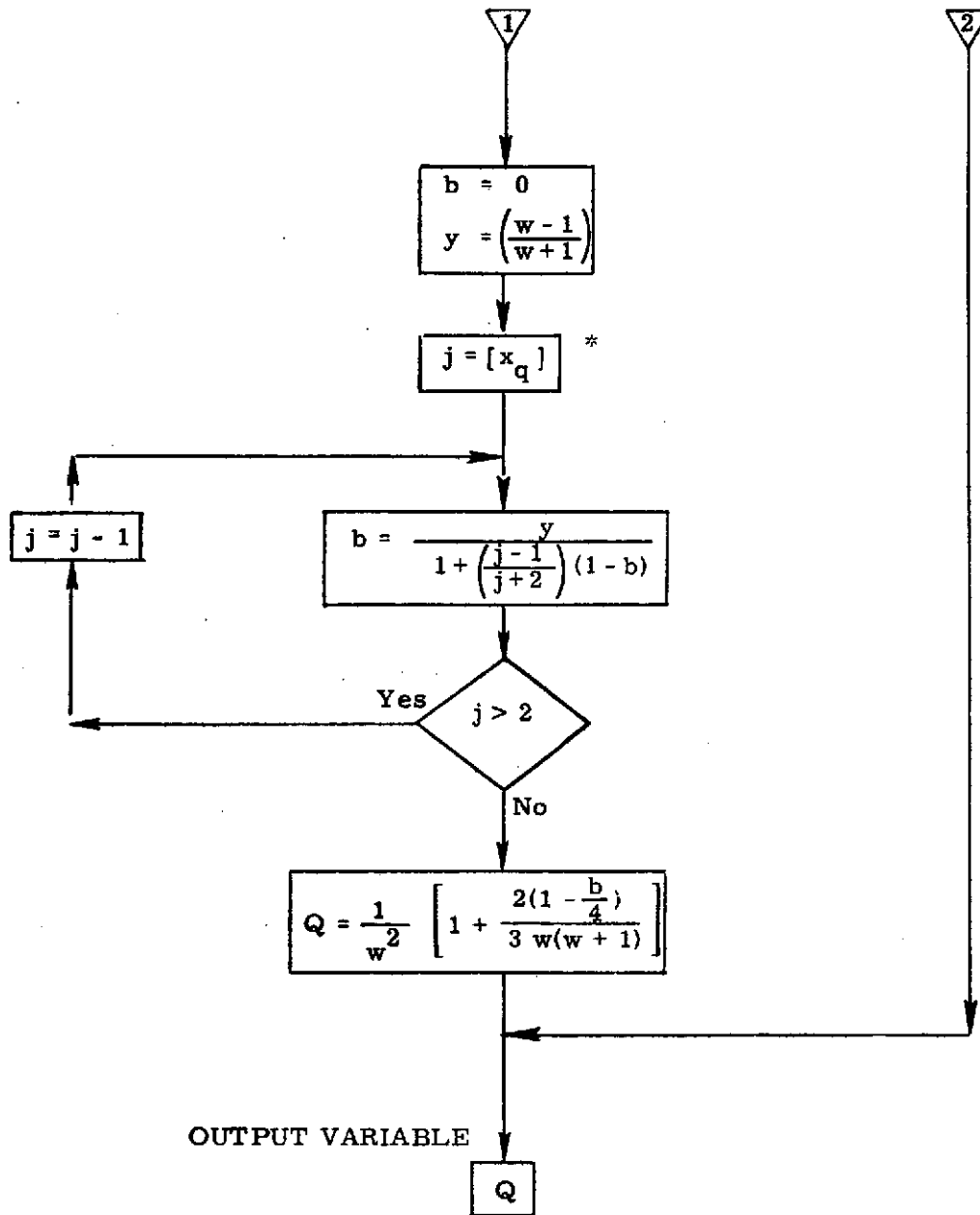


Figure 7a. Q Continued Fraction, Detailed Flow Diagram



\* The symbol  $[x_q]$  denotes the largest integer whose magnitude does not exceed the magnitude of  $x_q$ .

Figure 7b. Q Continued Fraction, Detailed Flow Diagram

### 5.3.4 Kepler Iteration Loop Subroutine

This subroutine is similar, though not identical, to the iteration loops used in the Conic Required Velocity routines. All could easily be combined into one routine but have been diagrammed separately for purposes of clarity.

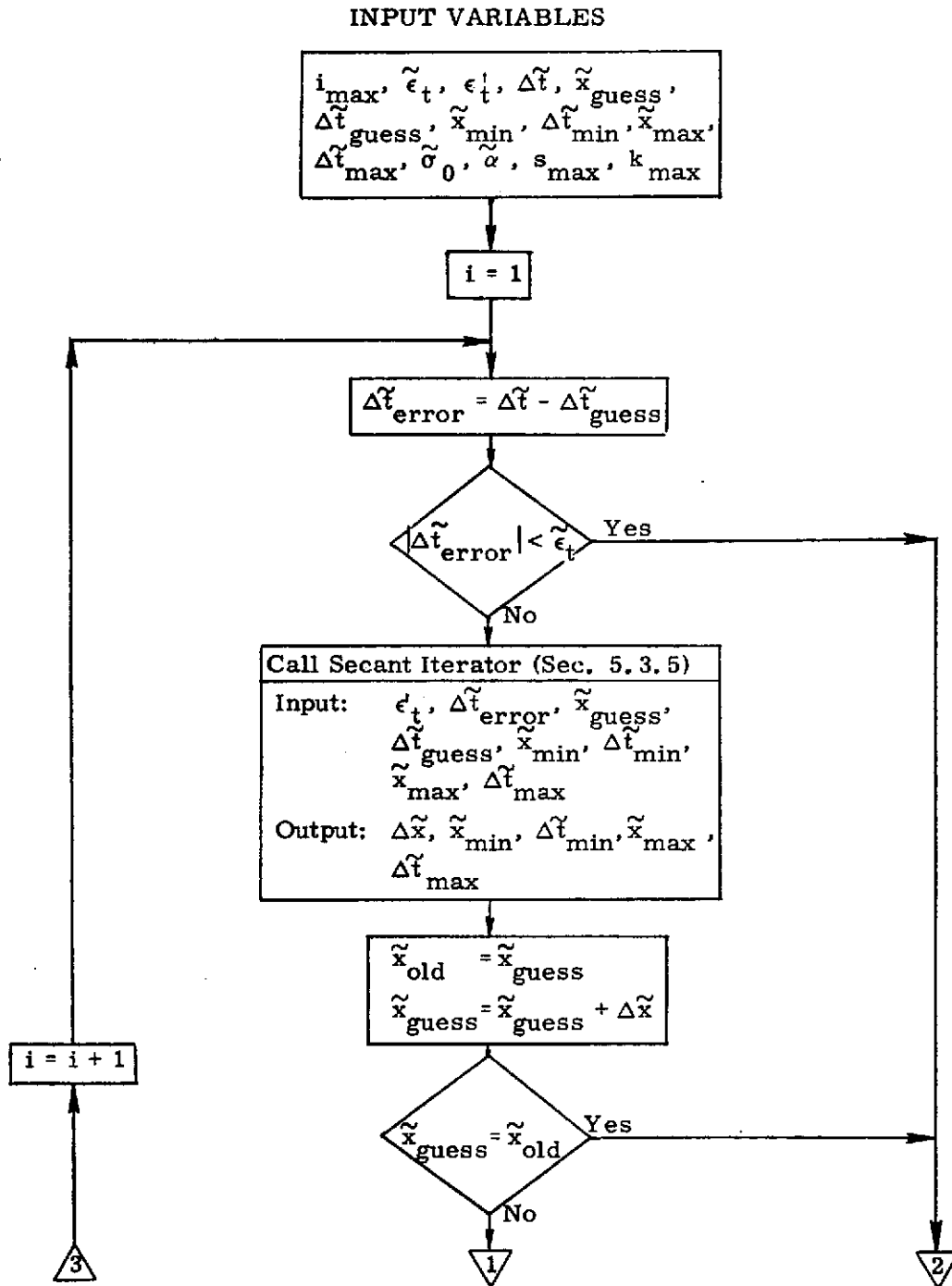


Figure 8a. Kepler Iteration Loop, Detailed Flow Diagram

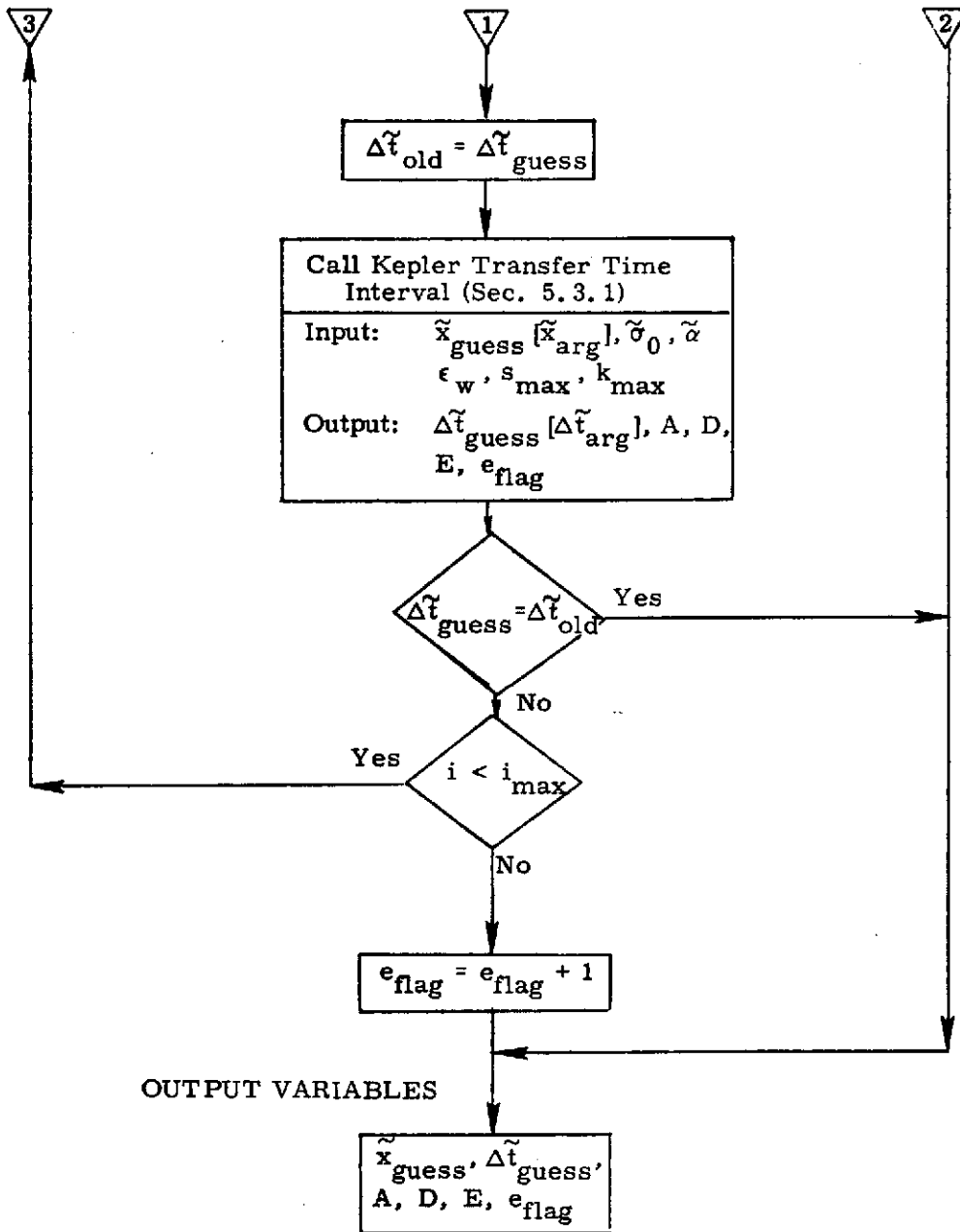


Figure 8b. Kepler Iteration Loop, Detailed Flow Diagram

### 5.3.5 Secant Iterator Subroutine

This subroutine is identical to the one used in the Conic Required Velocity routines.

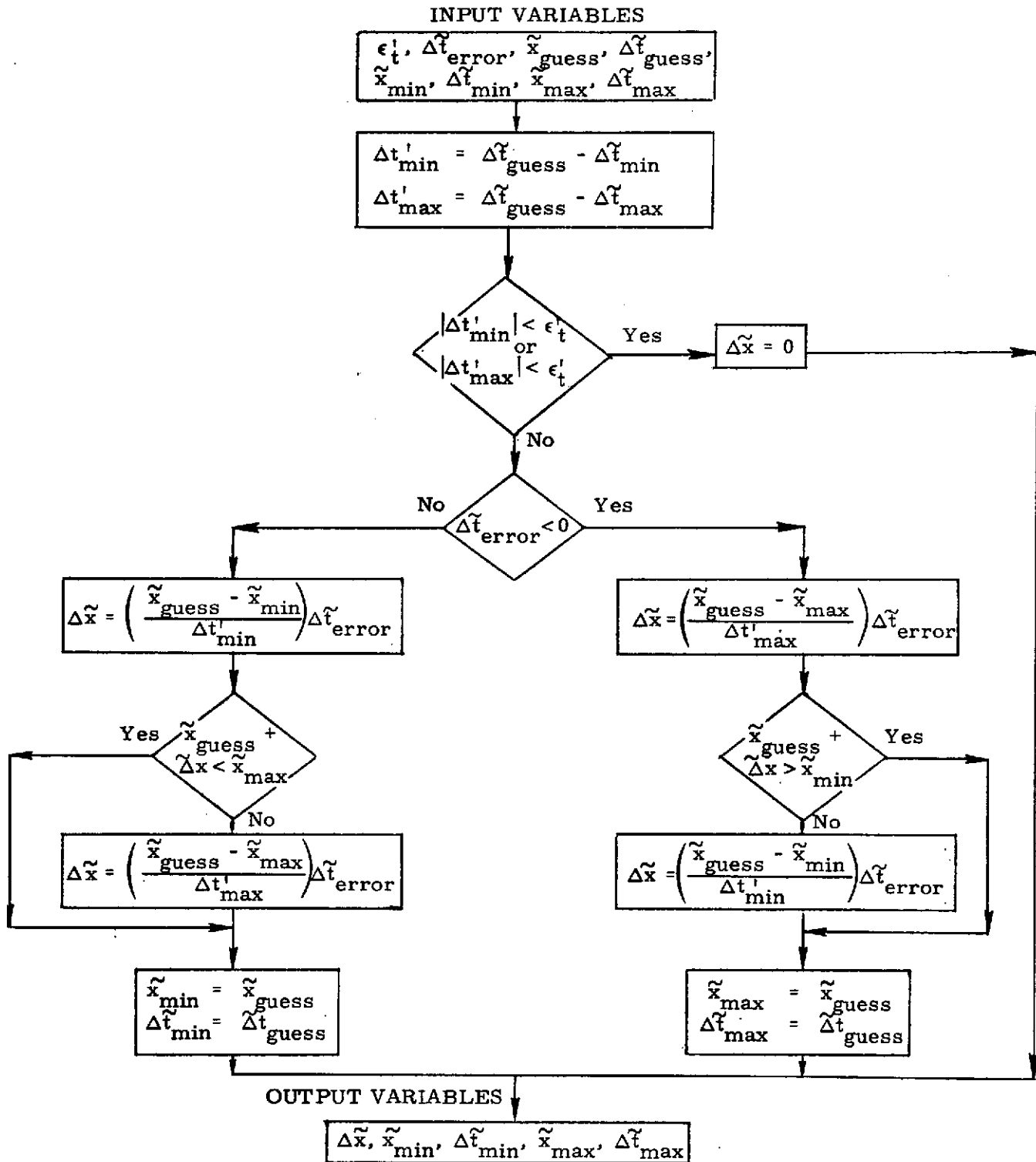


Figure 9. Secant Iterator, Detailed Flow Diagram

### 5.3.6 Extrapolated State Vector Subroutine

INPUT VARIABLES

$$f_2, b_4, \tilde{\sigma}_0, r_0, \\ A, D, E, \underline{i}_{r_0}, \tilde{\underline{v}}_0$$

$$F = 1 - 2A \\ \tilde{G} = 2(D E + \tilde{\sigma}_0 A) \\ \tilde{F}_t = -2 b_4 D E \\ G_t = 1 - 2 b_4 A$$

$$\underline{r} = r_0 (F \underline{i}_{r_0} + \tilde{G} \tilde{\underline{v}}_0) \\ \underline{v} = f_2 (\tilde{F}_t \underline{i}_{r_0} + G_t \tilde{\underline{v}}_0)$$

OUTPUT VARIABLES

$$\underline{r}, \underline{v}$$

Figure 10. Extrapolated State Vector, Detailed Flow Diagram

## 6. SUPPLEMENTARY INFORMATION

The analytic expressions for the generalized Kepler equation and the extrapolated position and velocity vectors are given in Battin (Ref. 3). Battin also devised the normalized formulation and organized the overall flow diagram structure. However, the details of the structure and the testing were done by Shepperd and Robertson.

A Newton iteration scheme may be devised at some time in the future. This would result in faster convergence near the solution point, but its worth depends on the expense of the derivative evaluation. It may be shown that if the derivative evaluation by itself takes more than 44% of the computation time used for the other calculations in one pass through the loop, then it is more efficient timewise to use the secant method.

## REFERENCES

1. Battin, R. H. , 1964, Astronautical Guidance, McGraw-Hill.
2. Battin, R. H. , and Fraser, D. C. , Notebook for Space Guidance and Navigation.
3. Battin, R. H. , and Fraser, D. C. , Space Guidance and Navigation, AIAA Professional Study Series.
4. Hildebrand, F. B. , 1956, Introduction to Numerical Analysis, McGraw-Hill.
5. Shepperd, S. W. , and Robertson, W. M. , June 1973, Conic Required Velocity, Draper Laboratory Space Shuttle GN&C Equation Document No. 26, to be published.