

The Generation of Numbers in Plato's *Parmenides*

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ABSTRACT

A difficult passage on metaphysical number theory in Plato's *Parmenides* is explained through reference to pre-symbolic systems of counting and calculation still common in the commercial environment of ancient Greece. The distinctions drawn between collections, numeric qualities, and number symbols explain both Plato's concept of number forms and Aristotle's failure to understand it. *Keywords:* Plato, *Parmenides*, numbers, arithmetic, calculation

In Plato's famously difficult *Parmenides*,² embedded deep within an investigation into the nature of the One and its relationship to the theory of Platonic forms, is a short passage (143b–144a) in which the eponymous Parmenides claims that he has demonstrated how all the numbers (meaning here the whole numbers greater than one) can be generated from first principles. As Aristotle indicates in the *Metaphysics*, however (987b), the process put forward leaves out the prime numbers, a fact that Parmenides is inexplicably unaware of, or is aware of but equally inexplicably fails to note.

This mysteriously obvious defect is perhaps the reason that the passage in question has been “almost unexplored by scholars.”³ Indeed, some just avert their eyes; in its exquisitely detailed discussion of this part of *Parmenides*, the *Stanford Encyclopedia of Philosophy* treats the most troublesome few lines as if they simply did not exist.⁴ This is easy to do; they are a digression from the main concern of this part of the dialogue, which is to show that the One is infinitely divisible. But I will demonstrate in what follows that these lines do describe a process that includes the missing primes and that an understanding of this process significantly clarifies Plato's theory of numbers.

¹ I am indebted to Clara Bosak-Schroeder and Serafina Cuomo for their encouragement and direction and to C. W. Marshall for his insights and early technical assistance. The positions taken here are solely my own.

² No less an authority than Gilbert Ryle (“Plato's *Parmenides* (II),” 305) called the second part of *Parmenides* “insufferably tedious.” Proclus, on the other hand (1.7), called it “nothing else than the celebrated generation of the Gods, and the procession of every kind of being from the ineffable and unknown cause of wholes” (Taylor, 20). Cherniss (“The *Parmenides* of Plato,” 126) sums up the entirety of *Parmenides* 142b–155e thus: “If the one is, then the one is everything.”

³ Călian, “One, Two, Three,” 49

⁴ Rickless, “Plato's *Parmenides*”

1 The mystery of the missing primes

The passage of interest in this paper follows a demonstration (*Parmenides* 142d-143a) that the one is not really one thing but two things—the one and its existence—and since the same is true of each part, this regress leads to an infinite multiplicity. (The inherently binary nature of this process will become relevant further on.) Beginning midway through 143a, the passage continues:⁵

[P1] We say that the one partakes of being and therefore it is? (Yes.)

[P2] And in this way, the one, if it has being, has turned out to be many? (True.)

[P3] But now, let us abstract the one which, as we say, partakes of being, and try to imagine it apart from that of which, as we say, it partakes—will this abstract one be one only or many? (One, I think.)

[P4] [143b] Let us see:—Must not the being of one be other than one? For the one is not being, but, considered as one, only partook of being? (Certainly.)

[P5] If being and the one be two different things, it is not because the one is one that it is other than being; nor because being is being that it is other than the one; but they differ from one another [ἕτερα ἀλλήλων] in virtue of otherness [ἄλλο] and difference [τῷ ἐτέρῳ]. (Certainly.)

[P6] So that difference⁶ [τὸ ἕτερον] is not the same—either with the one or with being? (Certainly not.)

Just as when we thought we had one existing thing, we really had two—that thing and its existence—now we have another thing, the *difference* between the thing and its existence, and all of these put together make three, as *Parmenides* goes on to demonstrate:

[P7] [143c] And therefore whether we take being and difference, or being and the one, or the one and difference, in every such case we take two things, which may be rightly called both. (How so?)

[P8] In this way—you may speak of being? (Yes.)

[P9] And also of one? (Yes.)

⁵ I have used here a slightly modified version of the Jowett translation, with each exchange labeled for later reference and the responses of *Parmenides'* *pro forma* interlocutor put into parentheses to set it off from the rest. Translations of short, unproblematic passages from familiar ancient sources cited in what follows have been taken from the Perseus collection (perseus.tufts.edu) or from a Loeb edition (often the same text). Where not individually credited, all the translations should be understood to represent well-known work not my own.

⁶ From this point on in the passage under discussion, Jowett translates “difference” [τὸ ἕτερον] as “the other.” For clarity, I have followed Cornford’s 1939 translation (*Plato and Parmenides*, 140) by changing “the other” to “difference.”

[P10] Then now we have spoken of either of them? (Yes.)

[P11] Well, and when I speak of being and one, I speak of them both? (Certainly.)

[P12] And if I speak of being and difference, or of the one and difference—in any such case do I not speak of both? [143d] (Yes.)

[P13] And must not that which is correctly called both, be also two? (Undoubtedly.)

[P14] And of two things how can either by any possibility not be one? (It cannot.)

[P15] Then, if the individuals of the pair are together two, they must be severally one? (Clearly.)

[P16] And if each of them is one, then by the addition of any one to any pair, the whole becomes three? (Yes.)

[P17] And three are odd, and two are even? (Of course.)

A couple of things have happened here: two is generated from one by doubling, and three is generated by the conjunction of one and two.⁷ It is also noted that two is the first even number and three is the first odd number (the Greeks did not consider one to be a number,⁸ for reasons that will become clear shortly).

Now comes the part that has continued to puzzle readers from Aristotle to the present:

[P18] [143e] And if there are two there must also be twice, and if there are three there must be thrice; that is, if twice one makes two, and thrice one three? (Certainly.)

[P19] There are two, and twice, and therefore there must be twice two; and there are three, and there is thrice, and therefore there must be thrice three? (Of course.)

⁷ One way to understand the metaphysical journey from one to three is to think of The One (unity itself) giving rise to a “one” (an individual) that is actually the unity of a pair, because no real object can be conceived of without distinguishing it from everything that is *other* than that object; to think of a thing is necessarily to think of it in opposition to what is not that thing, and in this way the being of any individual thing implies *difference* or *otherness*. As Plotinus remarks, “a particular one ... is not one—for it is already many by being a particular one” (*Ennead* VI.2.10). Thus we have the One that is truly one and the one that is actually a two that has the unity of a conjoined pair, a “both,” like a pair of twins or a pair of scissors (this is echoed by the grammar of the Greek dual number in the original). Plotinus says that Parmenides in Plato “distinguishes from each other the first One, which is more properly called One, and the second which he calls ‘One-Many’ and the third, ‘One and Many’” (*Ennead* V.1.8), i. e., one, two, and three. According to the author of the notes collected as *The Theology of Arithmetic* (traditionally attributed to Iamblichus), “each thing and the universe as a whole is one as regards the natural and constitutive monad in it, but again each is divisible, in so far as it necessarily partakes of the material dyad as well. Hence the first conjunction of monad and dyad results in the first finite plurality...” (Waterfield, 41).

⁸ “The one is not a number.” Aristotle *Metaphysics* 1088a6, quoted by Plotinus in *Ennead* VI.2.10.

[P20] If there are three and twice, there is twice three; and if there are two and thrice, there is thrice two? (Undoubtedly.)

[P21] Here, then, we have even taken even times, [144a] and odd taken odd times, and even taken odd times, and odd taken even times. (True.)

[P22] And if this is so, does any number remain which has no necessity to be? (None whatever.)

[P23] Then if one is, number must also be? (It must.)

The last sentence (P23) states the finding that this exercise was intended to establish, and the main track of the dialogue goes on from there. It is the sentence immediately preceding it that has continued to trouble investigators.

Parmenides is saying at P22 that the process described here generates *all* of the numbers *from the first three*. The assertion that we've now covered all possible cases (even times even, odd times odd, even times odd, and odd times even—echoing the traditional categories of number in Greek mathematics⁹) appears to imply that through multiplication based on 2 and 3 we can create any number we like, beginning with 4, 6, and 9 by way of example (P19–P20). Ryle summarizes the conventional understanding of these lines: “As multiplying consists in, e.g., taking couples three at a time, or threes twice at a time, we can get any number in this way. All arithmetical concepts are automatically generated...”¹⁰ But, as Aristotle correctly observed, this procedure leaves out the (infinitely large) set of prime numbers.¹¹

Given Plato's well-known fascination with mathematics, it seems strange that he would have made such an elementary error or, alternatively, have failed to note the omission of the primes, an issue that rears its head as soon as the numbers 4 and 6 have been mentioned (isn't something missing in between?). Consider, for example, *Laws* 746d–747b, where we are told that the competent ruler will understand the singular utility of the number 5040, which can be divided up (without remainder) no fewer than 59 different ways. In *Laws* 771a–c it is further noted that the divisors of 5040 include all the numbers from 1 through 12 except 11, and 11 can for practical purposes be finessed by just ignoring a remainder of 2 ($5040 = 11 \times 450$ with 2 left over, and 450 is itself rich in useful divisors). These are not the observations of someone

⁹ Euclid, *Elements* Bk. 7, defs. 8–11. Euclid defines all four of these cases, but some translators mistakenly omit the fourth one on the grounds that “even times odd” and “odd times even” are the same thing, thus retrojecting our modern concept of multiplication as a commutative operation ($ab = ba$) into an ancient concept of number that considered the two cases to be fundamentally different. Nicomachus, for example, devotes separate chapters to “even times odd” and “odd times even” (*Introduction to Arithmetic*, Bk. I, Ch. 9 and 10). His definitions do not align perfectly with Euclid's, but his treatment leaves no doubt that the terms “even times odd” and “odd times even” were not considered to be interchangeable.

¹⁰ “Plato's Parmenides (II),” 305

¹¹ Actually, more than the primes are omitted here: the supposed formula also leaves out all of the composite numbers that have prime factors greater than 2 or 3, beginning with 10, 14, and so on.

ignorant of factorization and the basic operations of arithmetic, and dismissing the problem by claiming that the missing primes don't matter¹² is less than satisfying.

No better is the kind of attractive but simplistic solution proposed by F. M. Cornford:

The objection that prime numbers cannot be obtained by multiplication is invalid, since Plato evidently includes addition and starts with that when he *adds* one term to another to make two, and two to one to make three.¹³

If this interpretation were correct, P18–P21 and the notion of multiplication would be entirely irrelevant and those lines entirely meaningless, thus trading one puzzle for another.

More recently, D. Blyth has argued that the numbers referred to in the passage under discussion are ordinals, not cardinals. According to Blyth, Plato's theory of number is that "numbers are originally ordinal, a sequence of forms differentiated essentially by position in sequence so as to constitute the natural integers."¹⁴ There are several problems with this position; to begin with, there is no historical evidence for the proposition that ordinals came before cardinals. J. J. Cleary notes that "the predominant conception of number both in the Pythagorean and Euclidean traditions is that of a cardinal rather than an ordinal," pointing in particular to the earliest Greek system of numeric notation, the acrophonic, which "was only suitable for representing cardinal numbers, and was used mainly to record weights, measures and sums of money."¹⁵ The ordinals, on the other hand, were at this stage still being written out fully as number words—that is, in a form impossible to work with mathematically. The ordinals are primarily linguistic entities; they are numbers with all of their useful mathematical properties stripped out, leaving only sequence. As Blyth himself notes,¹⁶ ordinal numbers not only fail to support arithmetic operations, they cannot even support the primitive relational predicates "greater than" and "less than." Blyth's theory proposes that the most useless kind of number was the one to be adopted first. And the fact that ordinals convey only sequence conflicts with Aristotle's complaint that the Platonist generation of numbers does not put them in order of priority and posteriority.¹⁷

In fact, the Greeks themselves were quite clear on their original definition of number, and in that definition will ultimately lie the answer to our puzzle.

¹² "Scholars often point out that this argument fails to generate all the numbers, because it ignores the primes. This doesn't matter for the argument, since the aim is simply to produce an unlimited number of entities, and that is achieved even without the primes." Ross, *Parmenides*, 78

¹³ *Plato and Parmenides* 141n2 (italics in original)

¹⁴ "Platonic Number," 24

¹⁵ "Aristotle's Criticism," 5

¹⁶ "Platonic Number," 36

¹⁷ *Nicomachean Ethics* 1096a

Numbers as collections

The traditional Greek notion of number is summed up by Euclid at the beginning of *Elements* Book 7 in Definition 2: “A number (ἀριθμός) is a multitude (πλήθος) composed of units.” Aristotle says that “a number is a multitude of monads” (ὁ δ’ ἀριθμὸς πλήθος μονάδων).¹⁸ According to Iamblichus, the first definition of number was that of Thales, who called number a “whole composed of units” (μονάδων σύστημα) “following the Egyptian view”¹⁹ and a unit as “one of a group.”²⁰ Eudoxus considered number to be a “determinate multitude” (πλήθος ὀρισμένον),²¹ a definition echoed by Nicomachus.²²

As some modern commentators have observed,²³ this kind of number-as-a-collection is very close to what we would call a *set*. Indeed, the only significant differences between the oldest sense of number captured in Thales’ definition and a set are that the Greeks would not have accepted a set with just one member as a “multitude” (this is why one in this context is not a number) and that they would have considered our modern “empty set” to be a contradiction in terms.

It is essential to understand and bear constantly in mind that the ur-number defined here is *the collection itself* conceived of as a unity, not the number that would be arrived at by *counting up* the items it contains. In *Theaetetus* 198c, counting is defined as “considering how great any number in question is,” which clearly indicates that the number in question is something that exists before it is counted.

It will be important in the following discussion to hold firmly to the fact that this kind of number does not *represent* a collection of objects; it *is* the collection of objects, as when we say “a number of men entered the room” or “a leader was chosen from among their number.”

Numbers as qualities

Along with the idea of number as a collection—a noun, as it were—comes the idea of quantity as a *property* of a collection—number in an adjectival role. This is the role that “three” plays when we speak of three sides of a triangle or three persons of the Trinity or three plums left in the bowl. In each case, “three” describes the one characteristic (“threeness”) that is exhibited by all collections of three objects, the *shared property* of all collections that happen to contain just that many things. And the test of whether a given collection shares this

¹⁸ *Metaphysics* 1053a

¹⁹ Heath, *History of Greek Mathematics*, 69

²⁰ D’Ooge, *Introduction to Arithmetic*, 12

²¹ Heath, *History of Greek Mathematics*, 70; Cleary, *Aristotle’s Criticism*, 5n8

²² *Introduction to Arithmetic* I.7

²³ Including Klein, *Greek Mathematical Thought*, 51 (“Numbers are, in short, many units”); Burnyeat, “Platonism and Mathematics,” 167 (these numbers “are not the natural numbers as modernly conceived but sets of units”); Wedberg, *Plato’s Philosophy of Mathematics*, 74 (number is “a plurality of objects considered relative to a chosen unit concept”).

particular property is whether its elements can be associated one-for-one with another set known to have that property, with no elements left over or unaccounted for.²⁴ Three in this sense is the number defined by all the sets that have three elements, i.e., by this property of quantity that they all share.

Critically for our discussion, the property of threeness can be *directly apprehended*. Any ordinary human being, and many animals, can directly perceive the twoness or threeness of a collection of objects. This capability, dubbed “subitizing” in a 1949 article by E. L. Kaufman et al.,²⁵ has its limits; for ordinary humans, the ability to directly apprehend the quantity of objects in a completely unorganized collection ends at about five or six. This natural upper limit to human subitization can be extended somewhat through training and pattern recognition; for example, the conventional patterns of pips on playing cards extends subitization to a practical limit of ten.²⁶ This upper limit to direct number perception is a bit higher in us than in some other animals, showing that it is relative to our intelligence and information-processing capabilities; from the fact that we *can* directly and reliably apprehend a specific property of quantity in relatively small collections it follows that a hypothetical being with greater powers could directly apprehend this property in numbers of a larger size. In other words, the size of the collection the quantity of which can be directly apprehended is in theory unlimited, even though we humans happen not to be able (even with visual aids) to immediately grasp the quantity of items in a collection of more than ten just by looking at it. And within our human limits, *the property of quantity inherent in any collection is just as real as any other physical property that we can directly perceive*.

Quantity is not the only property of a collection that can be directly perceived. Another property of certain collections is that their elements can be arranged in a certain shape. For example, collections of 4, 9, 16, and 25 objects all share the property that they can be arranged in the shape of a square, and for this reason such collections are called “square numbers.” There are also “triangular” and other geometrically arranged numbers. The Pythagoreans created an entire mathematics of such special collections in the form of dot-diagrams or “figured numbers.”²⁷ The important point here is that properties such as “square” are directly perceivable physical properties of *collections themselves*.

Number symbols

The original notion of a number as a collection of objects with certain properties that can in theory be directly apprehended stands in sharp contrast to the later concept of a *number*

²⁴ This definition of numerical equivalence is known as Hume’s Principle. The mathematical term for one-to-one correspondence is *bijection*.

²⁵ “Discrimination of Visual Number,” 520. The term, suggested by C. C. Coulter, derives from “the classical Latin adjective *subitus*, meaning *sudden*, and medieval Latin verb *subitare*, meaning *to arrive suddenly*” (ibid.).

²⁶ See also G. A. Miller’s 1956 essay “The Magical Number Seven, Plus or Minus Two.”

²⁷ Heath, *A History of Greek Mathematics*, vol.1 76–84

symbol. The number symbol 3 is not the directly apprehensible property of all collections of three things; rather, it is that element of a canonical series of symbols 2, 3, 4, 5, 6, ... that comes after the symbol 2 and before the symbol 4.²⁸ Users of number symbols are provided with a set of rules for manipulating them; for example, the rules say that the string of symbols “2+2” can be replaced with the symbol “4.” *Counting* in the sense we understand today is the operation of associating the objects in a collection one-to-one with these symbols, in order, until each object has been associated with exactly one symbol, with the name of the last symbol being taken as the cardinality of the collection. We now have *arithmetic*, which comes from ἀριθμεῖν, “to count.”

Note that there is no actual *apprehensible property* inherent in these symbols (aside from the shapes of the symbols themselves), and their association with real sets of objects, which is in fact irreducibly mysterious, seems obvious only because we are accustomed to making the leap.²⁹ Indeed, the relationship between collections of objects on the one hand and number symbols on the other is so far from obvious that it can be difficult to teach children to understand it—to provide them with what Nicomachus called “ladders and bridges”³⁰ between the material world and the abstract symbol.

We often bridge the two concepts by saying that the number symbol *represents* the collection, and that an operation like addition *represents* the joining together of two collections, but this is too simple. In fact, number symbols represent certain *concepts*, and exploring the relation between these concepts and actual quantities would lead us into very deep water indeed. Suffice it to say that the *symbolic* operation of addition is not the same as the *physical* operation of combining two collections of objects. As G. Frege pointed out, arithmetic addition “does not in general correspond to any physical relationship.”³¹ The meaning of “+” is just that statements such as “2+2=4” are true statements. In short (as Frege put it), “+” is not “and.”³²

²⁸ The idea that numbers begin at two carried over into abstract numbers as well. Nicomachus (*Introduction to Arithmetic* II.7) defined “linear numbers” as “all those which begin with two and advance by the addition of one.” The doctrine that all numbers originate from one and that the first of these is the number two appears as late as 1478 in the *Treviso Arithmetic*, which states that “number is a multitude brought together or assembled from several units, and always from two at least, as is the case with 2, which is the first and smallest number.... Numeration is the representation of numbers by figures.... Of these, the first figure, 1, is not called a number but the source of number.” Swetz, *Capitalism and Arithmetic*, 41. A passage in *Hippias Major* (302a) is often translated in a way that appears to be saying that one is the first *odd* number, but in fact the word for “number” does not appear there, and the sense seems to be that one is odd but not a number.

²⁹ For further regarding the inherent strangeness of the association between numbers and physical reality, see Wigner, “The Unreasonable Effectiveness of Mathematics.”

³⁰ *Introduction to Arithmetic* I.3

³¹ *Foundations of Arithmetic* 14 (§9)

³² *Foundations of Arithmetic* 50 (§38)

In what follows it will be necessary maintain an appreciation for the radical difference between the concepts of number as collection and number as symbol. In symbolic addition, to add 1 to 1 is to *change* the 1 we start with into a new number, 2. But when one *thing* is put together with another *thing* (the basic idea of addition), they are not replaced by some new thing and they are not changed; they stay exactly what they were before they were placed together. Because they are together, we can now say that we have two of them because together they participate in twoness, but that does not change the things themselves. This is a fundamental difference between numbers of things on the one hand and number symbols on the other.

The Neoplatonic philosopher Plotinus was still maintaining this distinction six centuries later in his commentary on *Parmenides*.

For it was not the one which became two, neither the one which was added nor the one it was added to, but each of them remains one, as it was; and the “two” is predicated of both, but the “one” separately of each, which remains what it is.³³

[I]n so far as you are counting and numbering you are making something quantitative, but in so far as the underlying realities are two and each of them is one, if each one is an essential completion of the substance and unity is in each, you are speaking of a different and substantial number.... For you certainly do not make number here below when you go through one after another things which have their own existence and do not come together in the numbering; for what difference does it make in substance to one man if he is counted along with another?³⁴

This is the meaning of P14–P15 in the passage under discussion.

2 Accounting before symbols

In decoding the meaning of any ancient text we must have some regard for its historical context, and in this respect, it will be essential to appreciate that in classical Greece, the philosophers engaged in what we would now call numerical analysis (i.e., who were concerned with operations on numbers represented by symbols or geometric quantities in something like the way we do today) constituted a vanishingly small proportion of the idle educated. Indeed, the Euclidean style of proof was itself something very new in Plato’s time, the first example appearing around 430 BCE.³⁵

The historical fact is that the vast majority of Plato’s contemporaries, including much of his intended readership, could not perform the operations of elementary symbolic arithmetic. In *Theaetetus* (196a) we are told that many Athenians, asked to give the sum of 5 and 7, would

³³ *Ennead* VI.6.14

³⁴ *Ennead* VI.6.16

³⁵ For background to these statements and further details regarding the status of formal mathematics in ancient Greece, see Asper, “Two Cultures of Mathematics” and Cuomo, *Ancient Mathematics*, 39–50.

answer 11, and in *Hippias Minor* (366c–d) we learn that only a skilled arithmetician could be trusted to correctly multiply 3 times 700.

In such a context, the generic or default concept of “number” would be a combination of the first two senses described above: number-as-a-collection and number-as-a-property-of-a-collection.

Tokenization

We are so accustomed to managing assets through writing and arithmetic computations that it can be hard to imagine how an ordinary person could manage even a flock of sheep or the items in a storeroom without knowing the corresponding symbols representing their quantity, but in fact accounting without written symbols was a standard practice for millennia.

Archaeology over the past few decades has demonstrated how this was accomplished; in what is surely one of the greatest intellectual accomplishments of the human race, the management of collections of objects was effected by creating *physical proxies* of those collections, using *tokens* such as pebbles or variously shaped pellets of clay to represent the items making up the actual collection.³⁶

Consider the ancient owner of a flock of sheep. He wishes to send part of the flock to his brother for sale in town, but how can he guarantee that none go missing under the care of his servant? The solution: as each sheep in the collection passes by, he rolls up a pellet of clay, and when all the sheep have passed, he wraps the pellets in a clay *bullae* or envelope. He marks the *bullae* with his seal and sends it on with the sheep (there are a number of these *bullae* in the archaeological record, a few of them still unopened). When the sheep arrive in town, his brother reverses the process by breaking open the envelope and associating each pellet with a sheep as it is put into his pen. If the sheep and the contents of the envelope match (this is one-to-one correspondence again), none has gone missing.

The same system can be used to preserve inventories and legal records as sets of tokens kept in small containers. A property owner might say, for example, “This set of tokens, in a *bullae* with my seal, certifies the sale of this number of sheep from my flock” (note again that “number” here refers to an actual set of objects) or “The set of tokens in the yellow container shows how many jars of honey are left down in the storeroom at the moment; as you can see, we have plenty.” This level of accounting, which is more than adequate for local inventory management, is well established in Mesopotamia by the fourth millennium BCE, and there is some evidence for the use of tokens for this purpose as far back as the eighth.

Tokenization obviates the cognitive challenge of learning to count and to master a shared system of notation. Arithmetic calculation isn’t worth the trouble it takes to learn until societies need to begin controlling wealth on a large scale, and it is reasonable to assume that a token-based “popular arithmetic” long persisted among the largely innumerate and illiterate general population into Plato’s day and well beyond.

³⁶ Where not otherwise credited, the archaeological data given here and in what follows come from Nissen, Damerow, and Englund, *Archaic Bookkeeping*, 11–12 and 125–30.

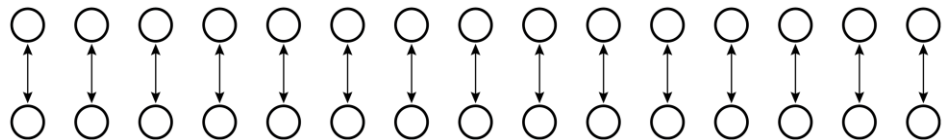
Proto-arithmetic

As we know from set theory, interesting mathematics can be practiced without number symbols, the heavy lifting provided (as in the examples above) by the operation of one-to-one mapping. This process serves not only for checking quantitative equality, but also, by simple extension, for exactly doubling a collection of any size or exactly dividing it in half (or exactly in half with a remainder of one) *without ever counting it*.

Consider, for example, this collection of objects:



There are too many here for their number to be directly apprehended by humans, but without needing to count we can still create another collection with exactly the same number of elements by associating each member of the first set with exactly one member of the second:



And the same operation can be used to check the equality of two halves. Thus the two innumerate brothers in the example above could exactly divide a flock of indeterminate size between them (perhaps with an odd sheep left over to provide a celebratory meal) without ever actually knowing the number symbol that would have been associated with either the original flock or its two halves.³⁷

Doubling through direct comparison is not arithmetical multiplication. To adopt the term used for this stage of development by Nissen et al.,³⁸ this is a *proto-arithmetical* kind of mathematics.

Token grouping

The fact that as humans our native untutored and unaided ability to apprehend number directly reaches a limit in sets of five or possibly six objects means that ways had to be found to conceptually grasp the relationships between greater quantities without knowing their counted number; quantity must be apprehended at a glance. In practice, this was accomplished by grouping collections of tokens to form larger units. We know (again from the archaeological record) how this was done: rules were developed, for a long time on an individual, ad hoc basis, whereby a certain small number of tokens (small enough for their

³⁷ The same direct comparison will, of course, establish which of two collections is the larger, again without counting anything. Thus we already have the operators “greater than” and “less than” as soon as we have the original concept of number-as-collection.

³⁸ *Archaic Bookkeeping*, 125ff

number to be apprehended directly) could be exchanged for a single differently shaped or sized token representing that group of units.³⁹

Given differently sized tokens representing one unit and five units, for example (such as a one-cent piece and a five-cent piece), we can maintain our conceptual grasp of the number of a set of fifteen (which is beyond what we can distinctly perceive) by reducing our set of fifteen tokens to just three. In Homer, *πεμπάζειν* “to five” meant “to count” (*Odyssey* 4.412),⁴⁰ and tallying by fives remains a common practice to this day. It seems plausible that the familiar pattern of two fives that we find at the ends of our arms was how we got to the direct perception of a group of ten; and this principle can be seen clearly in the archaic Greek acrophonic system of notation, whose “numerals” (representing 1, 5, 10, 50, 100, 500, 1000, 5000, 10000, and 50000) form a perfect stack of alternating 5×1 and 2×5 groupings. Roman numerals show the same principle of organization (I, V, X, L, C, D, M).

It would be a mistake, however, to attribute to early token grouping anything like this kind of polished and standardized regularity. An analysis of Sumerian clay tablets (which directly inherited some features of the earlier token systems) from a single site, Uruk, yielded thirteen distinct symbolic grouping systems, each dedicated to the management of a particular kind of good (measures of fish, dairy products, various kinds of grain, etc.). Some of these systems show groupings of 60 based on the sexagesimal notation that had evolved by that time, but others retain at their lower levels the smaller numbers associated with direct perception; for example, one system of notation consisted of a symbol for four units, then a differently shaped symbol for two of those four-unit groups, then a different symbol for two of *those* groups, and finally a fourth kind of symbol for ten of the last kind.⁴¹ Each of these systems used the principle of token grouping in different ways tuned to the requirements of particular trading and management contexts, details of which are now lost to us, but the great majority exhibit relationships among the smaller units (the ones used every day) that are well within the limits of direct subitization.

That there was a time before people had learned to form token groups was still remembered by Nicomachus, who noted that the use of a single symbol to represent a number greater than

³⁹ A typical system using tokens of different sizes to represent quantity and different shapes to indicate the kind of good referred to can be seen in Schmandt-Besserat, “Tokens: Their Significance,” which also provides a capsule summary of the history.

https://sites.utexas.edu/dsb/files/2014/03/tokens_article.pdf

⁴⁰ According to Plutarch (*Isis and Osiris* 374a), the Egyptians “speak of counting as ‘numbering by fives.’” Cf. *Moralia* 387e and 429d–f.

⁴¹ *Archaic Bookkeeping*, 25–29. Traditional (pre-metric) systems of weights and measures the world over exhibited similar kinds of directly perceptible small-number relationships between units and subunits (often, as in these ancient examples, maintaining different unit relationships for different trade goods) right up to their replacement in the 19th century. For an exhaustive compilation of traditional systems, see Martini, *Manuale di Metrologia*.

one was “by convention and agreement, not by nature,” whereas the natural way to write 1, 2, 3, 4, etc. would be 1, 11, 111, 1111, and so on.⁴²

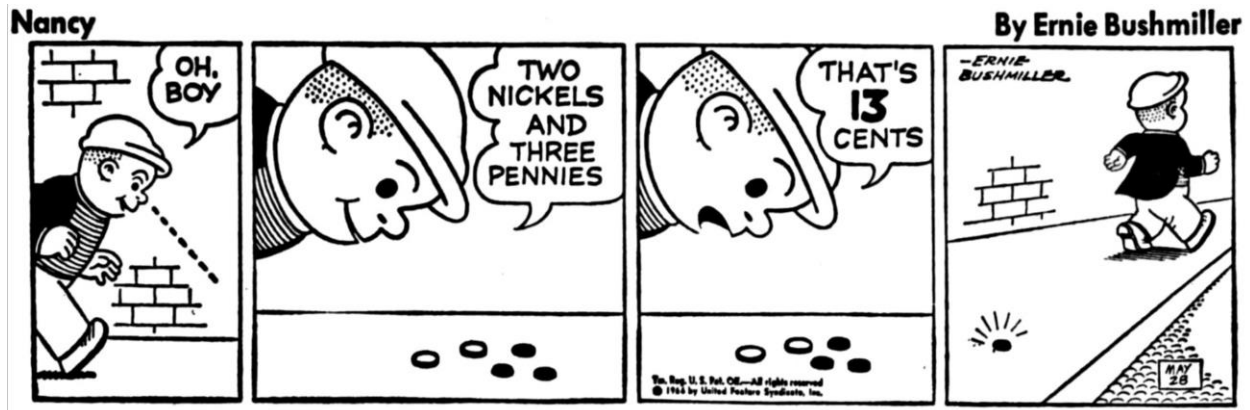


Figure 1. Sluggo demonstrates the difference between token grouping and counting. Note that token grouping comes first.

People and proxies

It can be difficult with our modern conception of integers strung out upon a conceptual number line to appreciate how deeply the ancients identified number with physical proxies, in particular, pebbles used as tokens. A character in a comedy by the early Greek playwright Epicharmus asks (DK 23B2) “If you took an odd number (ἀριθμός), or an even if you like, and chose to add or take away a pebble (ψήφος), do you think it would stay the same?”⁴³ Clearly at this point in time a number is still being thought of as an actual collection of token pebbles, an image that must have been familiar to everyone in the playwright’s audience. In Aristophanes’ *Wasps* (656–62), a character is told that a calculation is so simple that he can do it on his fingers instead of resorting to the use of pebbles. In explaining why all people count up to ten and then start over (with 10 plus 1, 10 plus 2, etc.) the Aristotelian *Problems* (15.3) observes that in having fingers, everyone carries his own set of pebbles. The word “count” in a line from Aeschylus’ *Agamemnon* (570) that Smyth translates “Why should we count the number of the slain” is actually ἐν ψήφῳ λέγειν.⁴⁴ We can even perceive a description of token grouping in terms of pebbles in this passage from Diogenes Laertius:

He [Solon] used to say that those who had influence with tyrants were like the pebbles employed in calculations; for, as each of the pebbles represented now a large

⁴² *Introduction to Arithmetic* 2.6. Using the Greek notation of that time, Nicomachus wrote this as $\alpha, \alpha\alpha, \alpha\alpha\alpha, \dots$

⁴³ The change here is not just (or even primarily) a change in quantity, but necessarily and more directly a change from odd to even or vice versa.

⁴⁴ The association between pebbles (ψήφοι) and numbers is also testified to by a number of common Greek expressions, e.g., λογίζεσθαι ψήφοις “to calculate,” καθαρὰι ψήφοις “exact accounts,” etc.; and of course our word “calculate” and its relatives derive from the Latin word *calculus* “a small stone.”

and now a small number, so the tyrants would treat each one of those about them at one time as great and famous, at another as of no account.⁴⁵

R. Netz has argued persuasively that *all* the ordinary calculations of classical Greek commerce were accomplished by representing numbers as tokens. Consider, for example, this description of how two collections represented by tokens could be added together using the widely attested abacus of the Greeks, which was not (like Asian versions) beads on strings but simply a set of inscribed lines, with each token on a line representing a multiple of 2 or 5 times the tokens on the line immediately above:

On the western abacus, movement is between lines, based on the definitional equivalences between numbers. Five times ten is fifty, and therefore five counters on the “ten” line are equivalent to a single counter on the “fifty” line; further, twice fifty is a hundred, and therefore two counters on the “fifty” line are equivalent to a single counter on the “hundred” line. Let us say, then, you start with four counters on the “ten” line and a single counter on the “fifty” line, and that you wish to add ten. You add a single counter to the “ten” line, and have now five counters there; the rules allow you now to remove those five, and to exchange them for a single counter on the “fifty” line. Now you have two counters on the “fifty” line; the rules allow you now to remove them, and to exchange them for a single counter on the “hundred” line. Here you stop, since no rules allow you to remove counters any longer, and so the calculation is complete: $90+10=100$. This is essentially all there is to it.⁴⁶

Note that nothing is being *counted* here in the sense we’re familiar with (a process of finding the last symbol in the numeric series to be associated with a collection of objects); rather, it is the *counters* (tokens) that are doing the counting, and the count at any point in the process just *is* the physical state of those tokens. It is only at the end that this state needs to be translated into symbols so that it can be recorded. At no point during this process are numbers the abstract entities of arithmetic; they are at every moment *physical proxies* of actual collections. It is this anciently common (albeit to us rather strange) concept of number that must be kept in mind here.

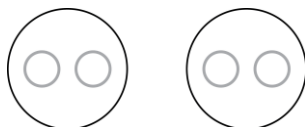
3 Proto-arithmetical generation

We are now in a position to understand P19–P20 in our passage from *Parmenides*, which seemed at first to be describing the (incomplete) generation of numbers by arithmetical multiplication using only the factors 2 and 3. It can now be seen that these lines have nothing to do with multiplication but instead set forth the principle of token grouping—that is, the method by which numbers are kept within the range of direct apprehension—using two and three (the first two numbers) as foundational examples.

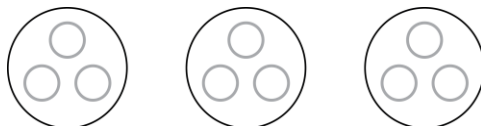
⁴⁵ *Lives* 1.2.59

⁴⁶ Netz, “Counter Culture,” 326 (italics in original)

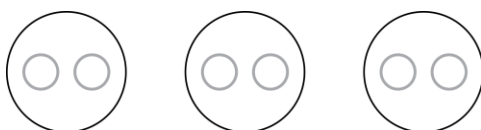
This will perhaps be clearer if shown in pictures. According to P19, the number four can be reduced to two groups:



And the number nine (which in a disorganized group cannot be apprehended directly) can be reduced to three:



In P20, the number six can be reduced either to three groups of two:



or two groups of three:



Thus (P21) “we have even taken even times, and odd taken odd times, and even taken odd times, and odd taken even times.” This is not multiplication, and it’s now clear why the products 4, 6, and 9 supposedly implied in P19–P20 appear nowhere in the text. It also explains why six is shown two different ways: this is to demonstrate two different approaches to *grouping*—the products of *multiplication* would be identical.

Proto-arithmetical operations

The preceding leaves unexplained the claim that all the numbers can be generated from the first three, but at least it is clear now that the mathematics in this part of *Parmenides* is proto-arithmetical, and the numbers being referred to in the passage under discussion are actual collections of objects or their token proxies, not number symbols. The next step in this analysis, then, is to inquire after the *proto-arithmetical mathematical operations* being demonstrated here (aside from one-to-one correspondence and token grouping, as previously discussed). It should be understood that such operations are no less mathematical than those of ordinary arithmetic; they just don’t yet involve *abstract* quantities. Our passage from *Parmenides* relies upon two such proto-arithmetical operations to generate all of the numbers (all possible sizes of collection), including the primes.

The first operation, demonstrated at length in the section (142d) immediately preceding the passage under discussion here, is *doubling* (each division of the whole into two parts doubles the number of entities). As pointed out earlier, the ability to exactly double or halve a collection of any size comes free with one-to-one correspondence; it is not yet an *arithmetic* operation.

It is hard to overstate the mathematical power of doubling and halving through direct comparison. As a number symbol, two is governed by the same arithmetical rules that govern any other abstract number, but in a proto-arithmetical context two is unique, because a multitude can be doubled just by creating another multitude whose equality with the first is verified by directly comparing them. It is not too much to say that where we have direct comparison, there we have duality.⁴⁷ It is no accident that ancient Egyptian arithmetic was based on doubling and halving,⁴⁸ and even today “the mental arithmetic of non-literate artisans and traders is often based on doubling, halving, and quartering.”⁴⁹

Even and odd

The second proto-arithmetical operation on display in *Parmenides* requires a bit more explanation. Basically, the conjunction of one and two gives birth to three, not as a process of arithmetical addition, but as a process of *making even into odd*.

Strange as it may seem, for the Greeks, the most important characteristic of a discrete number was its parity (evenness or oddness). Elusive as Plato can be in many of his references to mathematics, it is clear that he regards parity as an essential property of number.

In *Theaetetus* 198a, “the art of arithmetic” is described as “a hunt after the kinds of knowledge, or sciences, of all odd and even numbers.” In *Republic* 510c we are told that “students of geometry and reckoning and such subjects first postulate the odd and the even and the various figures and three kinds of angles and other things akin to these in each branch of science.” Regarding people who divide numbers into arbitrary groups or populations arbitrarily into races, it is said (*Statesman* 262e) that “a better division, more truly classified and more equal, would be made by dividing number into odd and even, and the human race into male and female.” And in *Epinomis* 990c (mistakenly attributed to Plato but certainly of his school) studies of number are said to be singularly concerned with “the whole origin of the odd and the even, and the greatness of their influence on the nature of reality.” Entirely absent from these passages (which already constitute a significant portion of everything Plato has to say about discrete numbers) is any mention of counted quantity, and it is clear that “number” here can only mean a collection of units.

The view of parity as the most important property of number was not unique to Plato. Nicomachus called odd and even “the most fundamental species” of number, “embracing the essence of quantity” and “reciprocally woven into harmony with each other, inseparably and uniformly by a wonderful and divine nature.”⁵⁰ According to Aristotle, the Pythagoreans

⁴⁷ Thus Plotinus: “[T]he double [cubit], in so far as it is called double, has its origin and its existence in comparison with the single cubits-length, and, without anything before this entering the mind, is called and is double in being compared with something else” (*Ennead* VI.3.21).

⁴⁸ Rossi, “Mixing, Building, and Feeding,” 409; Gillings, *In the Time of the Pharaohs*, 18–20; Heath, *History of Greek Mathematics*, 52–3

⁴⁹ Chrisomalis, “Foundations of Numbers,” 503

⁵⁰ *Introduction to Arithmetic* I.6 (D’Ooge translation)

placed “odd and even” second on the list of ten opposites that they considered to embody the basic principles of reality, just below “limit and the unlimited.”⁵¹ The Pythagorean list of opposites aligns “odd” with “male” and “even” with “female,” echoing the passage from *Statesman* quoted above.⁵²

In *Gorgias* 451a–c Socrates says that the concern of both calculation and numeration is “with the odd and even numbers, and the question of how many units there are in each”; this says explicitly that numbers exist as even or odd before they are counted. A little farther on (*Gorgias* 453e) it is said that the work of a person “skilled in numeration” “deals with the amount of an odd or even number,” which again implies that odd and even numbers exist prior to counting them. Indeed, Knorr has demonstrated⁵³ that several important theorems regarding odd and even can be rigorously proven using physical tokens (the dot diagrams of the Pythagoreans), entirely without reference to counted quantity. And in Euclid’s proof of the proposition that “if as many odd numbers as we please be added together, and their multitude be even, the whole will be even” (*Elements* Bk. 9 Prop. 22), it is not a whole *counted number* (the abstract symbol arrived at by counting) that is proven to be even but rather the multitude itself (ἔστι δὲ καὶ τὸ πλῆθος τῶν μονάδων ἄρτιον).

In *Theaetetus* 185d it is asked “through what bodily organs we perceive by our soul the odd and the even,” which implies that this property can be directly apprehended, like the property of quantity, and this proves to be the case. Unlike subitization, which hits a practical ceiling around ten, the Greek way of defining “even” allows the direct perception of parity in collections of arbitrary size.

For the ancient Greeks, a number (that is, a collection of units) was *even* if it could be divided into two smaller collections of equal size with none left over. This corresponds exactly to the illustration of one-to-one correspondence shown earlier, in which it was demonstrated that equality can be judged independent of quantity; that image is just how an ancient Greek would think of (and indeed directly perceive) the evenness of a collection of objects. This perception of evenness clearly does not depend on knowing the counted number of units (i.e., the symbol that would be associated with the quantity in that collection). And the same is true in perceiving the oddness of an odd number.

⁵¹ *Metaphysics* 986a. Cf. Plutarch, *Isis and Osiris* 370e.

⁵² A passage in Plutarch’s *The E at Delphi* illustrates several themes here: “[E]very number may be classified as even or odd, and unity, by virtue of its potentiality, is common to both, for the reason that its addition makes the odd number even and the even number odd, and since two makes the first of the even numbers and three the first of the odd, and five is produced by the union of these two numbers, very naturally five has come to be honoured as being the first number created out of the first numbers; and it has received the name of ‘marriage’ because of the resemblance of the even number to the female and of the odd number to the male” (388a, Babbitt translation). The association of even with female is also noted in *The Theology of Arithmetic* (Waterfield 65, 143).

⁵³ *Evolution of the Euclidean Elements*, 140–2

When we think of adding one to an even number (thus making it an odd number) we generally think of moving one location farther down the sequence of number symbols; thus 12 becomes odd by adding 1 and making it 13. This operation is inextricably bound up with the concepts of *succession* and *arithmetic addition*, and it requires us to know the symbol associated with the even number we start with and the symbol associated with the odd number we will produce if we add 1 to it.

The Greeks, however, thought of this process as placing a new object *between* two equal groups of objects to make an even collection into an odd collection. If one visualizes the previous diagram showing the direct comparison of equal multitudes, and then imagines a new token placed between the two rows, it will be seen that the oddness of the resulting assemblage is just as directly perceivable and just as little dependent on arithmetic or the cardinality of the new collection as the perception of evenness was before the new unit was placed in the middle.

Thus, Nicomachus (I.7) says “that is *even* which is capable of being divided into two equal parts without a unit falling in the middle, and that is *odd* which cannot be divided into two equal parts because of the aforesaid intervention (*μεσιτείαν*) of the unit.”⁵⁴ A fragment of Aristoxenos tells us that “even numbers are those which are divided into equal parts, while odd numbers are divided into unequal parts and have a middle term,”⁵⁵ and Plutarch says that “the even, when parted in any direction, leaves as it were within itself...a field; but, when the same thing is done to the odd, there is always a middle left over from the division.”⁵⁶ The transition from even to odd is conceived of as a kind of completion; a passage going back ultimately to Poseidonios says that “When the odd is divided into equal parts, a unit is left over in the middle; but when the even is so divided, an empty field is left, without a master and without a number, showing that it is defective and incomplete.”⁵⁷

The idea of making the even odd by interposing a unit between two halves is apparently due to the Pythagoreans; Aristotle says of them (*Metaphysics* 1083b) that “in an odd number they make the Ideal One the middle unit.”

It can now be seen that the proto-arithmetical operation of *making the even odd* (and therefore completing it) by interposing an additional unit is the process described in P16–P17. The role of the one in this process appears to be what Plato is referring to in *Phaedo* 105c with the statement that instead of saying “oddness is the cause of the odd numbers” we should say that “the monad is the cause of them.”⁵⁸

⁵⁴ Heath, T. L., *The Thirteen Books of Euclid's Elements*, 281

⁵⁵ Burnet, *Early Greek Philosophy*, 212

⁵⁶ *Ibid.*

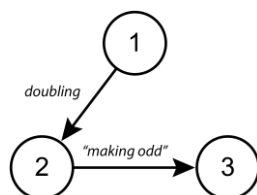
⁵⁷ *Ibid.*

⁵⁸ In an apocryphal letter of the emperor Julian we find the statement that “there is a saying of wise men of old that an odd number is to be preferred to an even, and they declare that the source of increase is that which does not couple” (*Works of the Emperor Julian*, 277).

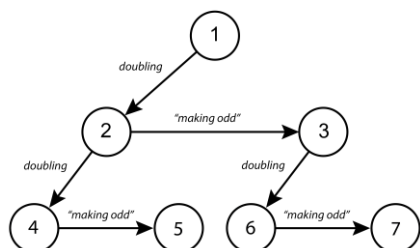
Generating the numbers

The claim in P22 is that *all* of the numbers (i.e., token representations of all of the theoretically directly apprehensible *collections of units*) can be generated from the first three using the same operations that created two and three from one, that is, doubling and “making odd.” And a trial shows this to be exactly the case.

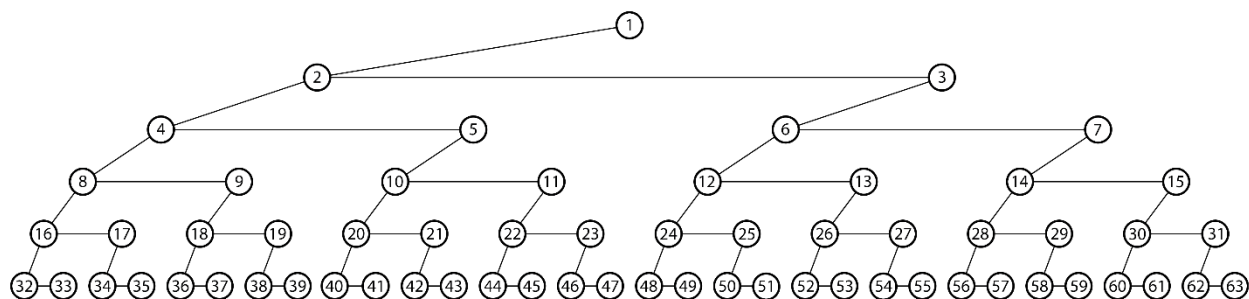
The first set of operations can be diagrammed thus:



Since the two operations of doubling and “making odd” can be applied to collections of *any size*, it follows that the two-step operation that creates two and three can similarly be applied to two to produce four and five, and to three to produce six and seven:



By recursively applying this two-part proto-arithmetical function at each node, we can populate the entire tree, the first six levels of which are shown here:



It is obvious by inspection that this process omits none of the whole numbers (realized here as conceptual piles of tokens), just as Plato said. And every operation in the construction of this tree can be *physically* carried out without counting, arithmetic, or any reference to numbers as we commonly understand them.

It is worth observing that every number in this diagram is a member (and possibly the source) of an infinitely long chain of doubling relationships,⁵⁹ and in this sense every number

⁵⁹ In considering the *generative power* of two, it is well to remember the a “even” with “female” and “odd” (which stands at the beginning of each chain) with “male.”

participates in the binary process of division described in *Parmenides* 142d. Indeed, K. M. Sayre (1983) comes up to the edge of the solution shown here when he observes that numbers written in modern binary notation and arranged in rows by number of digits (1 in the first row, 10 and 11 in the second row, 100, 101, 110, and 111 in the third row, and so on) will create exactly the tree exhibited above—a powerful demonstration of its inherent duality—but because Plato had no knowledge of binary notation, Sayre rejects this explanation and comes instead to the conventional conclusion that “Plato assumes the operation of multiplication and then projects the series of positive integers by successive application of this operation,” the missing primes being simply “an apparent oversight.”⁶⁰

If the explanation offered above is correct, there can be little doubt that the process of generation described in *Parmenides* 143b–144a is Pythagorean in origin; it corresponds at every point with the account of Pythagorean mathematics given by Alexander Polyhistor as reported by Diogenes Laertius: “The first principle of all things is the monad. From the monad there arises the indeterminate [or unlimited] dyad, which then serves as passive material to the monad, while the monad serves as active cause. From the monad and the indeterminate dyad there arise numbers...”⁶¹

4 Wider implications

Two important consequences for the study of Plato follow from the realization that our passage from *Parmenides* is concerned with numbers as collections (or their proxies) and the directly apprehensible properties of those collections. First, it clarifies Plato’s concept of number forms; and second, it explains how Aristotle misunderstood that concept.

Numbers and forms

A comprehensive discussion of Plato’s theory of forms is beyond the scope of this paper, so in what follows I will simply assume that the reader is already familiar with the general idea. The difficulties attending the relation between numbers and forms arise chiefly from Aristotle’s objections to the theory;⁶² what Plato himself had to say seems perfectly clear.

Let us return for a moment to quantity as a property that can be directly apprehended. Unlike number symbols, quantities that can be directly apprehended have each a unique *character*. The essential character of two differs from that of three as the essential dynamic of a *ménage à deux* differs from that of a *ménage à trois*. This character of each number is directly perceived along with the sense of the quantity itself; the character of two is felt as “balanced” and the character of three is felt as “dynamic” in the way that the character of a color or a melody is sensed as “happy” or “sad.” The ability of particular numbers to evince particular associations (the details of which are to some extent culturally determined) lies at the basis of arithmology

⁶⁰ *Plato’s Late Ontology*, 56

⁶¹ Diogenes Laertius 8 (Wheelwright trans. 224)

⁶² See Annas (*Aristotle’s Metaphysics* 13–19) for an extended treatment of Aristotle’s critique of Plato’s theory of number forms.

or “the theology of numbers,” and there can be little doubt that the association of specific qualities with particular numbers was inherited from the Pythagoreans.⁶³ Nothing like this character attaches to a number as a symbol that happens to come after one symbol in a particular series and before another.⁶⁴

According to Aristotle,⁶⁵ the Platonists of his time recognized three kinds of numbers: “sensibles” (τὰ αἰσθητά), “forms” (τὰ εἶδη), and “mathematicals” (τὰ μαθηματικά). Given the distinctions drawn earlier, it can now be seen that *sensibles* are numbers in the original sense, i.e., actual collections of objects whose quantity can (in theory) be directly apprehended as a sensible property; *mathematicals* are the abstract numbers;⁶⁶ and *forms* are those properties of the sensibles (not of the abstract numbers) that give each real-world collection its particular quantity and numeric *character*.

That Plato himself was aware of the differences between physical collections, numeric qualities (forms), and number symbols is testified in at least three places in his writings. The first is at *Theaetetus* 195e–196a, where Plato explicitly distinguishes “seven men and five men” (as the meaning his interlocutor *would naturally assume*) from the abstract numbers 7 and 5.

⁶³ *Metaphysics* 985b, 990a, 1078b. In his commentary on the *Metaphysics*, Alexander of Aphrodisias draws on a lost work of Aristotle to note several of these associations between numbers and concepts, for example that *justice* was held by the Pythagoreans to be the character of “the first square number,” i.e., four (*On Aristotle Metaphysics I*, 64–66). Such associations are ubiquitous and persistent; thus we have Philo, centuries later, writing in *De Mundi Opificio* (*Περὶ τῆς Μωυσεως κοσμοποιιας*) XVI that “the number four is the first number which is a square, being equal on all sides, the measure of justice and equality” (*Works of Philo Judaeus*, 13). Compare the figurative meanings of the English word “foursquare”: *firm, solid, honest*.

⁶⁴ The character associated with specific numbers seems to decrease in strength and increase in variability of interpretation as numbers increase in size, finally fading away entirely in numbers past ten. This is in keeping with the continuum of certainty in subitization, ranging from the universal ability to directly perceive two and three to the rather tenuous ability, with training and pattern aids, to directly perceive ten. Compare this with Aristotle’s claim that some of the Platonists ended the form numbers at ten (*Metaphysics* 1073a, 1084a; *Physics* 206b), which no doubt relates to the Pythagorean primacy of the decad (*Metaphysics* 986a). In *The Theology of Arithmetic*, Iamblichus calls the decad “a measure and a complete boundary of every number, and there is no longer any natural number after it, but all subsequent numbers are produced by participation in the decad...” (Waterfield 61) and says that the Pythagoreans called the decad “the all” “because there is no natural number greater than it, but... number recurs and circles back, in a sense, to the decad; for a hekatontad is ten decads, and a chiliad is ten hekatontads, and a myrad is ten chiliads, and similarly any other number recurs and retrogresses either to the decad or to some number within the decad. Anyway, the reduction and returning of all numbers to it is manifold” (Waterfield 110). In the same collection, Anatolius is quoted as saying that ten is “the limit and perimeter of all number” (Waterfield 114). See also Theon of Smyrna, *Mathematics Useful for the Study of Plato* 2.39 (Lawlor 66) and the Aristotelian *Problems* 15.3.

⁶⁵ *Metaphysics* 987b

⁶⁶ And geometric objects that lie outside the scope of this discussion.

The second can be found in *Phaedo* 101b–c, where the *form* of twoness (a perceptible quality of a collection consisting of two objects) is explicitly distinguished from the *arithmetic addition* (προστεθέντος) of one plus one. And in *Phaedo* 101b Plato refers to symbolic, proto-arithmetical, and formal concepts of number in the same sentence: “wouldn’t you be afraid to say that ten is greater than eight by two [arithmetic] and this is the reason it’s bigger, rather than by the size of the number [proto-arithmetical collection] and because of the form of number [its directly perceptible numeric character]?”⁶⁷

Once it is seen that Plato’s numeric forms are properties of numbers as *sets* and not properties of numbers as *integers*, confusion about how numbers relate to forms disappears. Within the context of Platonic forms, the relationship is perfectly straightforward: a set containing three objects participates in the form of “three” the way that a rose participates in the form of “rose,” and the notion of many different triplets of objects all participating in the form of threeness is no more problematic than the notion of many different roses all participating in the form of roseness. Indeed, the relationship between numbers and their forms is one of the more intuitively reasonable examples of Plato’s theory. Number symbols, on the other hand, have nothing to do with forms.

The wrong path

Some of Aristotle’s confusion regarding Plato’s theory of numbers arises from a failure to understand that it is collections of objects that contain units, not the number symbols used in counting them. (This is really just another form of the stubbornly persistent misapprehension that number symbols *represent* collections of objects.) The mistaken assumption that abstract numbers contain units leads to the further mistaken notion that the numeric *forms* must contain units. But nowhere in Plato’s writings does he refer to number forms as containing units; for Plato, forms are unique, simple, and indivisible, and the numeric forms are no exception.

More importantly, by Aristotle’s time, the proto-arithmetic mathematical operations that can be seen in early Pythagoreanism and as late as our fragment from *Parmenides* appear to have been replaced in philosophical discourse with something like our modern idea of arithmetic. In *Metaphysics* 1084a, Aristotle says that the Platonists generate all of the numbers beyond two through a combination of three processes: (a) “when one is added to an even number, we get an odd number,” (b) “when one is multiplied by two, we get ascending powers of two,” and (c) “when powers of two are multiplied by odd numbers, we get the remaining even numbers.”

Aristotle’s process (a) is the arithmetic version of what I have called “making odd,” and his process (b) corresponds exactly to what I have called “doubling”; and, as we have seen, the set-theoretic versions of these two primitive mathematical operations are sufficient to generate sets corresponding to all of the positive integers, including the primes. Aristotle’s belief that a third process is needed to generate all of the numbers (which even then, he says, omits the primes) shows that he—and, as seems likely, the followers of Plato from whom he received

⁶⁷ Trans. Emlyn-Jones and Preddy

this—misunderstood the generative process as arithmetical rather than proto-arithmetical, concerned with number symbols rather than with collections, and, on this mistaken reading, in need of a third ancillary operation to create (some of) the integers.

Blyth conjectures that “Speusippus, Plato’s successor at the Academy...having rejected the theory of forms as an account of numbers, might well have been forced to generate mathematical numbers by arithmetic operations,”⁶⁸ and (though Blyth does not prefer it) this seems to be a reasonable historical explanation for the existence of the superfluous third generative process in Aristotle’s account. Whatever the reason, the confusion of later arithmetical processes with proto-arithmetical mathematics seems to have led the entire further discussion of Platonic numbers down a blind alley.

One problem above all seems to have misled commentators regarding the passage we began with. It is easy to follow Cornford in assuming (as noted earlier) that in P16 “Plato evidently includes addition... when he *adds* one term to another to make two, and two to one to make three.” This reading is based on the misapprehension that the word translated as “addition” in P16 refers to Frege’s “+”, i.e., to the arithmetic operation of adding two numbers to produce a third number according to the symbol manipulation rules for addition. But a closer look shows the usual translation to be incorrect. The word conventionally rendered as “addition” in P16 is συντεθέντος (from συντίθημι), literally “having placed with.”⁶⁹ Together with its adjective and noun forms, συντίθημι appears no less than 83 times in the extant writings of Plato,⁷⁰ and aside from its occurrence in P16, it is never used by Plato to refer to the addition of two numbers but rather used in various non-mathematical senses of combining, composing, or agreeing.

In his other works, Plato always uses προστίθημι rather than συντίθημι to signify arithmetic addition. Examples include *Cratylus* 432b, *Phaedo* 96e, 97b, and 101b, and *Republic* 527a.⁷¹ The noun form, πρόσθεσις, appears in *Phaedo* 97a, 101b, and 101c with the specific meaning of arithmetic addition and appears with no other meaning anywhere else in Plato. This strongly suggests that arithmetic addition is not the operation being described in P16. The phrase “the

⁶⁸ “Platonic Number in the *Parmenides*,” 25

⁶⁹ I am indebted to C. W. Marshall for bringing this to my attention.

⁷⁰ As συντίθημι (combine, compose): *Apology* 27a; *Cratylus* 383a, 396a, 424e, 425a (×2), 427c, 433e (×2), 434a (×2), 434b (×2), 435a; *Crito* 52d (×3), 54c; *Epistles, Letter 7* 341b; *Euthydemus* 305c, 306a (×2); *Gorgias* 520c; *Laws* 669e, 669d, 812d, 816c, 920d; *Menexenus* 236b, 245c, 249d; *Phaedo* 60c, 77c, 78c (×2), 92b, 93a; *Phaedrus* 228a, 254d, 278c (×3), 278d; *Philebus* 29e; *Republic* 359a (×2), 359b, 377d, 400b, 533b, 618c; *Sophist* 219a, 252b (×2); *Statesman* 259d, 273b, 276e, 279d, 288e; *Theaetetus* 183c, 203c; *Timaeus* 54d, 69d, 72e. As σύνθετος (composite): *Laws* 722e; *Phaedo* 78c, 86a, 92a; *Philebus* 29e; *Republic* 381a, 400b; *Sophist* 219a; *Statesman* 279d (×2), 288e. As σύνθεσις (combination, composition): *Cratylus* 431c; *Phaedo* 93a; *Republic* 533b, 611b; *Sophist* 263d; *Statesman* 280b, 280c.

⁷¹ For later examples of προστίθημι as the usual word for arithmetic addition, see Sextus Empiricus, *Against the Physicists* I.326–7.

addition of any one to any pair” in P16 is literally “the combination of any one with any pair” and would more accurately be translated that way regardless of the theory presented here.

A little further on at *Parmenides* 148e–149c the numbers one, two, and three make a repeat appearance that is worth reviewing in detail.⁷² Note here their palpable *physicality*:

[148e] Must not that which is to touch another be next to that which it is to touch, and occupy the place nearest to that in which what it touches is situated? (True.) Then the one, if it is to touch itself, ought to be situated next to itself, and occupy the place next to that in which itself is? (It ought.) And that would require that the one should be two, and be in two places at once, [149a] and this, while it is one, will never happen. (No.) Then the one cannot touch itself any more than it can be two? (It cannot.) Neither can it touch others. (Why not?) The reason is, that whatever is to touch another must be in separation from, and next to, that which it is to touch, and no third thing can be between them. (True.)

The placement of the third thing *between* the first two echoes the placement of the unit that makes an even collection into an odd one. Whatever one, two, and three are in this passage, they are obviously not *symbols*, but some interpreters⁷³ have taken the part immediately following this to be a reference to arithmetic addition:

[149b] Two things, then, at the least are necessary to make contact possible? (They are.) And if to the two a third be added in due order, the number of terms will be three, and the contacts two? (Yes.) And every additional term makes one additional contact, whence it follows that the contacts are one less in number than the terms; the first two terms exceeded the number of contacts by one, and the whole number of terms exceeds the whole number of contacts by one in like manner; [149c] and for every one which is afterwards added to the number of terms, one contact is added to the contacts. (True.) Whatever is the whole number of things, the contacts will always be one less. (True.)

The first thing to notice here is that we are never concerned with the *abstract number* of either the collection of terms or the collection of contacts; the demonstration proceeds simply by comparing the collection of *things* one-for-one with the collection of *contacts* and noting that one thing is always left over. This exemplifies the kind of bijectional token-based thinking about numbers that was discussed earlier in connection with odd and even. More to the point, the word that is translated as “added to” is neither Plato’s usual προστίθημι for arithmetic addition nor the form of συντίθημι used in P16 to refer to physical adjacency but rather a third term, προσγίγνομαι, which literally means “attach to.” In the 28 times Plato uses this word aside from *Parmenides* 148e–149c,⁷⁴ it never means the arithmetic addition of two numbers

⁷² This is the Jowett translation again.

⁷³ In particular, an anonymous reviewer, to whom I am obliged for pointing to this instance.

⁷⁴ *Cratylus* 421b; *Epistles*, Letter 6 322c; *Hippias Major* 289d (×2), 289e, 290b, 292d; *Laws* 789a, 850a, 908b, 929a, 946c; *Parmenides* 153e; *Philebus* 37d (×2); *Republic* 346d, 375e, 437e, 438e, 609a (×2); *Sophist* 238a (×2); *Theaetetus* 206c, 210b; *Timaeus* 42e, 82b, 86e.

but rather the *combination* of one non-numeric thing with another. If for “added to” in the passage above we substitute the more literal “combined with” or “joined to,” we obtain a very different reading.

The fact that the words translated as “add” or “addition” in these passages from *Parmenides* are used nowhere else in Plato to mean arithmetic addition supports the thesis that “addition” in the passage we began with does not refer to our familiar number symbols and methods of symbol manipulation but belongs to an older layer of traditional proto-mathematical knowledge much closer to the practical management of physical goods.

5 The larger context

Parmenides 143b–144a sets forth a *metaphysics* of number, which (according to the interpretation offered here) uses the proto-arithmetical manipulation of actual objects as the operative metaphor. It would be helpful to see this located in the metaphysics of the dialogue as a whole; it does serve to establish that the existence of the one leads to the existence of the many, but this point has already just been made in the dialogue, and the mechanism of 143b–144a seems intended to mean something beyond that, or to function as a piece of some larger theory.

Plato himself claimed that he was relating at fourth hand an actual conversation that occurred when Parmenides and his student Zeno visited Athens in Socrates’ youth.⁷⁵ This has often been regarded as a fiction, but the claim that the young Socrates met with Parmenides is repeated twice elsewhere (in *Theaetetus* 183e and *Sophist* 217c), and it seems strange that Plato would fabricate out of whole cloth an event that would have been in living memory just a generation earlier. In any case, the writings of Zeno and Parmenides were still extant in Plato’s time (*Parmenides* 127c, *Symposium* 195c), and it is reasonable to assume that Plato is trying accurately to portray the teachings of the historical Parmenides in the way that he treats the doctrines of Zeno earlier in the dialogue. The One in Plato’s dialogue does in fact have the properties ascribed to Being by the historical Parmenides: “that it is in itself and the same as itself, that it is at rest, that it is like itself, that it is in contact with itself, etc.”⁷⁶ Thus it will perhaps not be inappropriate to look for the metaphysical ground of our passage in what remains of Parmenides himself. And in this context, our passage makes sense.

The lesson it contains is that the explosion of the one into the infinitude of the phenomenal cosmos occurs with the introduction of *difference* or *distinction* (P5). In two fragments from the historical Parmenides’ long poem quoted by Simplicius, the goddess tells Parmenides that human beings lost the truth (that the one is all that exists and that there is no non-being) when they

established two forms to name in their judgments,
of which it is not right to name one—in this they have gone astray—
and they distinguished things opposite in body, and established signs

⁷⁵ *Parmenides* 127b

⁷⁶ Palmer, “Parmenides”

apart from one another...⁷⁷

But since all things have been named light and night
and the things which accord with their powers
have been assigned to these things and those,
all is full of light and obscure night together,
of both equally, since neither has any share of nothing.⁷⁸

In other words, everything is now seen in pairs of opposites; the One that has no parts has given rise to the one that implies its opposite in order to exist, and this conceptual distinction between things that are inside a category and things outside of that category generates the whole universe of apparent objects and collections of objects. Difference entails comparison, comparison entails duality, and the combination of duality and the one engenders the world of things. According to the historical Parmenides, this multiplicity is an illusion, and the source of this illusion is the difference or otherness of opposite thought-forms. Thus while it is difficult to fit an extended interpretation of *Parmenides* 143b–144a into the larger context of Plato’s dialogue, it is fairly easy to fit it in with what we know of the teachings of the historical Parmenides and may even be a survival of some of those teachings.

Seeing our fragment as a survival of the historical Parmenides would also explain its metaphorical reliance on conceptions of number and numeric operations that were becoming archaic by Plato’s time. It has long been established that the Extinction Event for the original Pythagorean conception of numbers represented in dot-diagrams was the discovery of irrational numbers like the square root of 2 (the length of the diagonal of a unit square). This occurred during the life of the historical Parmenides, and it’s not difficult to imagine that his conception of number was still what he had inherited from centuries of practice before his time.

6 Conclusion

The interpretation of *Parmenides* 143b–144a provided in this paper, based on the observation that the default sense of “number” in Plato is (as Euclid said) a collection of objects and that Plato’s mathematics of number is often the mathematics of physical collections, may cause discomfort to those who have accepted a picture of Plato’s mathematics as based throughout on our modern symbolic conception of numbers and on other mathematical ideas more advanced than the proto-arithmetical concepts considered here. I suspect that this conventional picture has been influenced by the mentions of ratios and geometric objects in dialogues such as *Timaeus*, subjects that for us are strongly associated with numerical calculation. But according to Aristotle,⁷⁹ ratios (unlike our modern rational numbers) were not themselves considered to be numbers at all, and insofar as geometric objects can be

⁷⁷ Simplicius, *Commentary on Aristotle’s Physics* 145.1–146.25 in Curd, *A Presocratics Reader*, 61

⁷⁸ *Ibid.*, 180.9–12

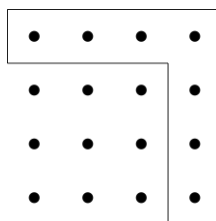
⁷⁹ *Metaphysics* 991b; cf. 981b.

associated with numbers, those numbers are continuous quantities, something like our real numbers, and not the discrete numerical quantities with which we are concerned here.

When Plato does refer to discrete quantities, on the other hand, those references can more often than not be understood in terms of collections; for example, the 59 different ways one can divide up 5040 is about physically dividing up collections of 5040 *things* (specifically phratries, demes, villages, military companies, etc.⁸⁰) into 59 different sizes of *subcollection*. Even instances of what appears to be true arithmetic addition (for example, *Parmenides* 154b and 154d) can often just as easily be read as referring to the physical combination of two things or the union of two sets.

That Plato's default conception of number was the same as that of Euclid seems unexceptionable, and we have known since the late nineteenth century that collections have their own mathematics independent of and *logically prior* to the mathematics of numbers as abstract symbols. More recent archaeology has demonstrated that calculations in Plato's time were carried out using physical proxies with methods that are *historically* prior to symbol manipulation as well. It hardly seems a stretch to imagine that Plato (or perhaps the historical Parmenides) was basing at least some of his theory of numbers on the everyday commercial mathematics of his time.

If there is a larger lesson here, it might be that we should try harder to understand ancient ways of thinking when interpreting ancient texts. Consider the gnomon of a square, as it featured in Pythagorean mathematics. If we have a square number (that is, a collection of things that can be arranged in a square, say three things on a side), and we add enough other things to make it the next larger square (for example, four things on a side), the L-shaped set of things that get added (in this case, seven of them) is called a *gnomon*. The Pythagoreans diagrammed this with dots, as shown here:



It is usual to generalize the behavior of the gnomon as we change the size of the square by saying that at each step up in size, the gnomon takes the next value in the sequence of odd numbers (1, 3, 5, 7, 9, 11,...), so that, for example, the next increase in the size of the square (from $4 \times 4 = 16$ to $5 \times 5 = 25$) will be attained this time by adding a gnomon of 9 rather than 7. Squares continue to grow by adding gnomons of 11, 13, 15, and so on. As Heath puts it,⁸¹ "if n^2 is any square number, the addition of the odd number $2n+1$ makes it into the next square, $(n+1)^2$." Thus we are given to understand a Pythagorean concept in algebraic terms.

⁸⁰ *Laws* 746d

⁸¹ *History of Greek Mathematics*, vol. 1, 77

But it's also possible to describe gnomons in a way that has nothing to do with abstract numbers: at each stage, the gnomon is generated by applying the proto-arithmetical “double-and-odd” operation described earlier to the units making up one side of the square. This is the token-based analogue of the algebraic term $2n+1$ in Heath's description, but without any reference to counting, arithmetic, symbols, numeric sequence, the calculated numeric value of the square, or for that matter any particular numeric value at all. Knorr⁸² proves no fewer than sixteen theorems about dot-diagrams like these entirely without reference to numbers in the modern sense.

Obviously not all of ancient mathematics can be approached in this way, but it might be well to avoid the assumption that Plato and the mathematicians of his time had entirely moved on from conceiving of numbers in physical terms when this manner of thinking had long been, and would remain for centuries to come, foundational to the mathematics of the marketplace and the storeroom.

⁸² *Evolution of the Euclidean Elements*

WORKS CITED

- Alexander of Aphrodisias. *On Aristotle Metaphysics I*. Translated by William E. Dooley. Bloomsbury: London and New York, 1989.
- Annas, J. (trans.). *Aristotle's Metaphysics Books M and N*. Oxford: Clarendon Press, 1976.
- Asper, Markus. "The Two Cultures of Mathematics in Ancient Greece." Chapter. In *The Oxford Handbook of the History of Mathematics*, edited by Eleanor Robson and Jacqueline Stedall, 107–32. Oxford and New York: Oxford University Press, 2010.
- Babbitt, F. C. (trans.). Plutarch *Moralia V*. Cambridge and London: Harvard University Press, 1936.
- Blyth, D. "Platonic Number in the *Parmenides* and *Metaphysics XIII*." *International Journal of Philosophical Studies* 8, no. 1 (2000): 23–45.
- Burnet, J. *Early Greek Philosophy*, 3d ed. London: A & C Black, 1920.
- Burnyeat, M. F. "Platonism and Mathematics: a Prelude to Discussion." Chapter. In *Explorations in Ancient and Modern Philosophy*, 2:145–72. Cambridge: Cambridge University Press, 2012.
- Călian, F. G. "One, Two, Three... A Discussion on the Generation of Numbers in Plato's *Parmenides*." In *New Europe College Stefan Odobleja Program Yearbook 2013–2014*, edited by Irina Vainovski-Mihai, 49–78. Bucharest: New Europe College, 2015.
- Cherniss, H. F. "Parmenides and the *Parmenides* of Plato." *American Journal of Philology*, Vol. 53 No. 2 (1932), 122–38.
- Chrisomalis, S. "The Cognitive and Cultural Foundations of Numbers." Chapter. In *The Oxford Handbook of the History of Mathematics*, edited by Eleanor Robson and Jacqueline Stedall, 495–517. Oxford and New York: Oxford University Press, 2010.
- Cleary, J. J. "Aristotle's Criticism of Plato's Theory of Form Numbers." Chapter. In *Platon und Aristoteles—sub Ratione Veritatis: Festschrift für Wolfgang Wieland zum 70. Geburtstag*, edited by Gregor Damschen, Rainer Enskat, and Alejandro G. Vigo, 3–30. Göttingen: Vandenhoeck & Ruprecht, 2003.
- Cornford, F. M. *Plato and Parmenides*. London: Routledge and Kegan Paul, 1939.
- Cuomo, S. *Ancient Mathematics*. London and New York: Routledge, 2001.
- Curd, P. and R. D. McKirahan (trans.) *A Presocratics Reader: Selected Fragments and Testimonia*. 2d ed. Indianapolis: Hackett Publishing, 2011.
- D'Ooge, M. L. (trans.) Nicomachus of Gerasa, *Introduction to Arithmetic*. New York and London: Macmillan, 1926.
- Emlyn-Jones, C. and W. Preddy (trans.) *Plato I: Euthyphro, Apology, Crito, Phaedo*. Cambridge and London: Harvard University Press, 2017.

- Fowler, H. N. (trans.) *Plato IV: Cratylus, Parmenides, Greater Hippias, Lesser Hippias*. Cambridge and London: Harvard University Press, 1939.
- Frege, G. *The Foundations of Arithmetic: A Logico-mathematical Enquiry into the Concept of Number*. Translated by J. L. Austin. Second revised edition. New York: Harper & Brothers, 1953.
- Gill, M. L. (trans.). *Plato Parmenides*. Indianapolis and Cambridge: Hackett Publishing, 1996.
- Gillings, L. J. *Mathematics in the Time of the Pharaohs*. Cambridge: MIT Press, 1972 (Dover reprint 1982).
- Heath, T. L. *A History of Greek Mathematics*. Vol. 1. *From Thales to Euclid*. Oxford: Clarendon Press, 1921 (Dover reprint 1981).
- Heath, T. L. *The Thirteen Books of Euclid's Elements*. Cambridge: Cambridge University Press, 1908.
- Julian. *The Works of the Emperor Julian*. Translated by W. C. Wright. London: William Heinemann, 1923.
- Kaufman, E.L. et al. "The Discrimination of Visual Number." *American Journal of Psychology* 62, no. 4 (1949): 498–525.
- Klein, J. *Greek Mathematical Thought and the Origins of Algebra*. Translated by Eva Brann. Cambridge: MIT Press, 1968 (Dover reprint 1992).
- Knorr, W. R. *The Evolution of the Euclidean Elements: A Study of the Theory of Incommensurable Magnitudes and Its Significance for Early Greek Geometry*. Dordrecht and Boston: D. Reidel, 1975.
- Martini, A. *Manuale di Metrologia, Ossia Misura, Pesi e Monete in Uso Attualmente e Anticamente Presso Tutti i Popoli*. Torino: Loescher, 1883.
- Miller, G. A. "The Magical Number Seven, Plus or Minus Two." *Psychological Review* 101, No. 2 (1956), 343–52.
- Nissen, H. J., P. Damerow, and R. K. Englund. *Archaic Bookkeeping: Writing and Techniques of Economic Administration in the Ancient Near East*. Translated by Paul Larsen. Chicago: University of Chicago Press, 1993.
- Netz, R. "Counter Culture: Towards a History of Greek Numeracy." *History of Science* 40, No. 3 (2002), 321–52.
- Palmer, J. "Parmenides." In *Stanford Encyclopedia of Philosophy*. Stanford University, 1997–. Article first published Fri Feb 8 2008; substantive revision Mon Oct 19, 2020.
- Rickless, S. "Plato's Parmenides." In *Stanford Encyclopedia of Philosophy*. Stanford University, 1997–. Article first published Fri Aug 17, 2007; substantive revision Tue Jan 14, 2020.
- Rossi, C. "Mixing, Building, and Feeding: Mathematics and Technology in Ancient Egypt." Chapter. In *The Oxford Handbook of the History of Mathematics*, edited by Eleanor Robson and Jacqueline Stedall, 407–28. Oxford and New York: Oxford University Press, 2010.

- Ryle, G. "Plato's *Parmenides* (II)." *Mind* 48, no. 191 (July 1939): 302–25.
- Sayre, K. M. *Plato's Late Ontology: A Riddle Resolved*. Princeton: Princeton University Press, 1983.
- Schmandt-Besserat, D. "Tokens: their Significance for the Origin of Counting and Writing." https://sites.utexas.edu/dsb/files/2014/03/tokens_article.pdf.
- Taylor, T. (trans.). *Six Books of Proclus on the Theology of Plato*. London: Printed for the author, 1816.
- Theon of Smyrna. *Mathematics Useful for Understanding Plato*. Translated by Robert and Deborah Lawlor. San Diego: Wizard's Bookshelf, 1979.
- Waterfield, R. (trans.) Iamblichus (attrib.) *The Theology of Arithmetic*. Grand Rapids, MI: Phanes Press, 1988.
- Wetz, F. J. *Capitalism and Arithmetic: The New Math of the 15th Century*. Text of the *Treviso Arithmetic* translated by David Eugene Smith. La Salle, Illinois: Open Court, 1987.
- Wheelwright, P. (trans.) *The Presocratics*. New York: Odyssey Press, 1966.
- Wigner, E. "The Unreasonable Effectiveness of Mathematics in the Natural Sciences." *Communications in Pure and Applied Mathematics*, Vol. 13, No. 1 (February 1960): 1–14.
- Yonge, C. D. (trans.). *The Works of Philo Judaeus, the Contemporary of Josephus*. Vol. 1. London: George Bell & Sons, 1800.