

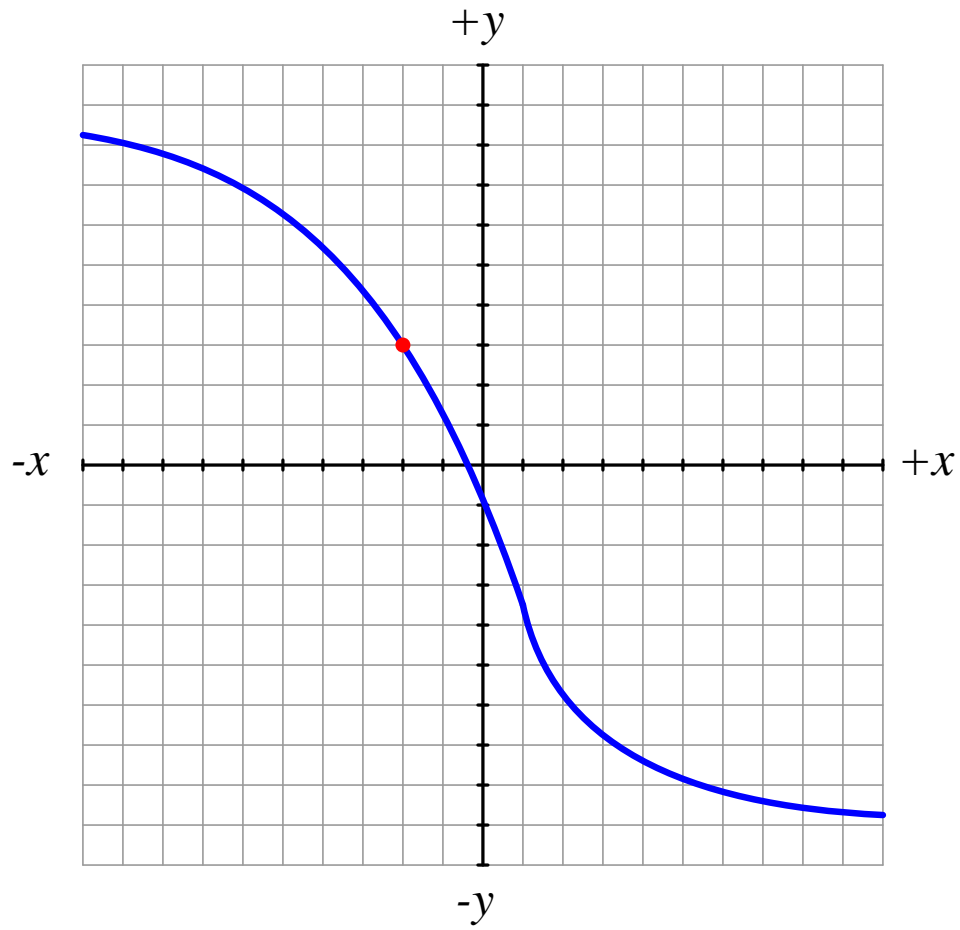
Calculus concepts and applications

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Questions

Question 1

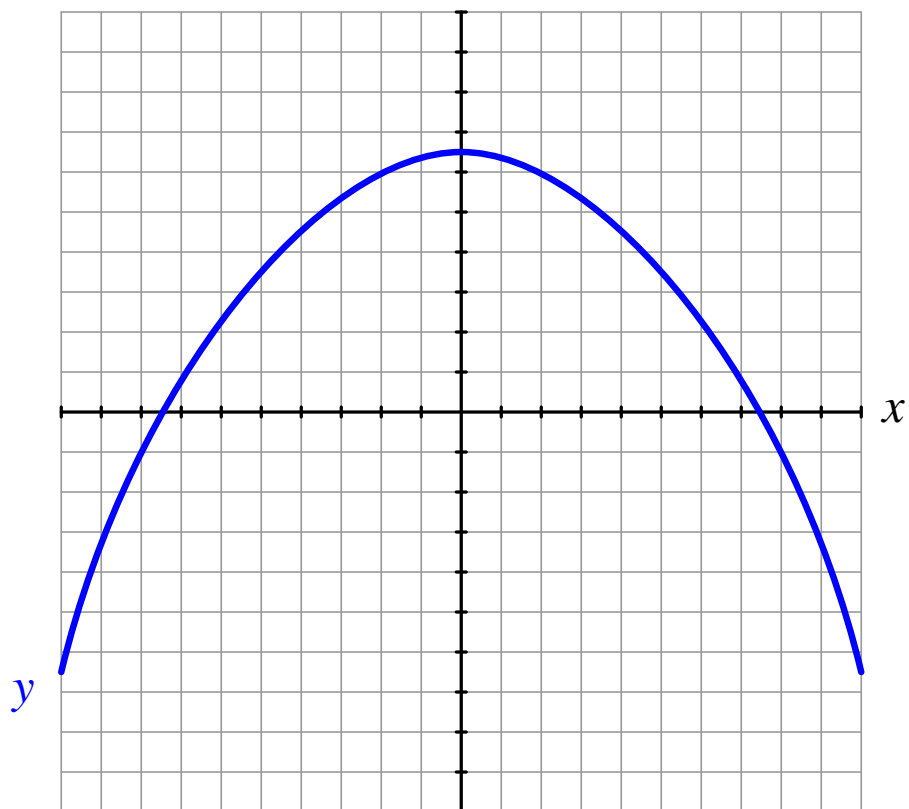
Sketch the graphical interpretation of the derivative $\frac{dy}{dx}$ at the point specified on the graph (the red dot), and determine whether its value is *positive* or *negative*:



file i01347

Question 2

Shade the area on this graph representing the following Riemann sum (assuming each horizontal and vertical division on the graph has an incremental value of 1):

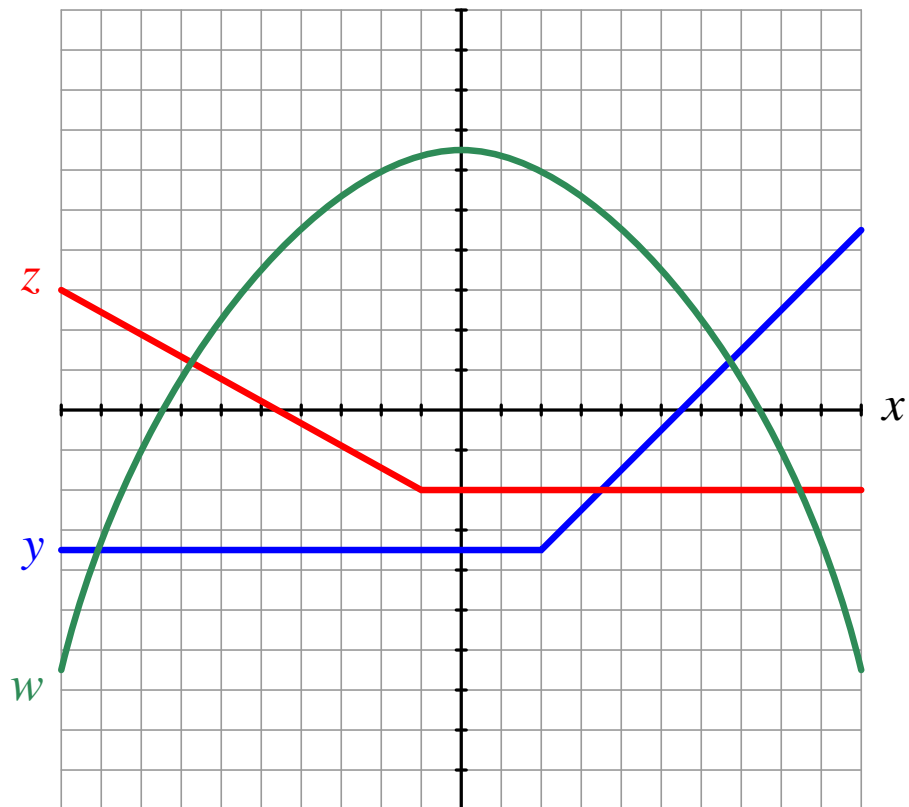


$$\sum_{n=-4}^9 y \Delta x_n$$

Also, determine whether the numerical value of this Riemann sum is *positive* or *negative*.
file i01921

Question 3

Shade the area on this graph representing the following integrals (assuming each horizontal and vertical division on the graph has an incremental value of 1):



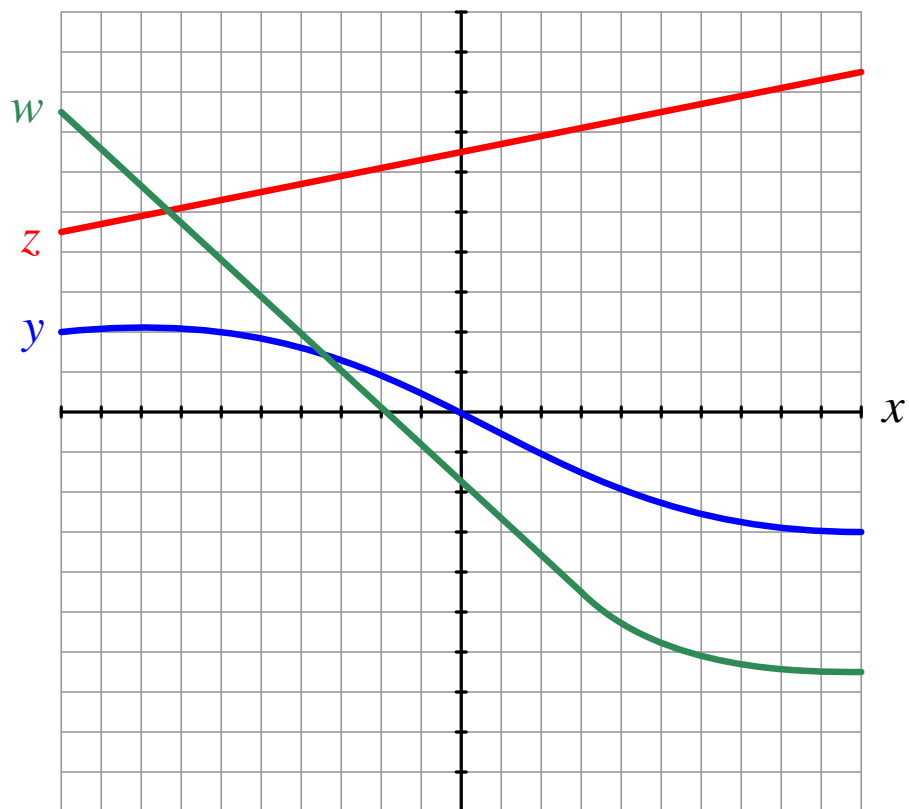
$$\int_0^5 y \, dx$$

$$\int_{-1}^{-6} w \, dx$$

Also, determine whether the numerical value of this integral is *positive* or *negative*.
[file i01345](#)

Question 4

Shade the area(s) on this graph representing the following integral (assuming each horizontal and vertical division on the graph has an incremental value of 1):

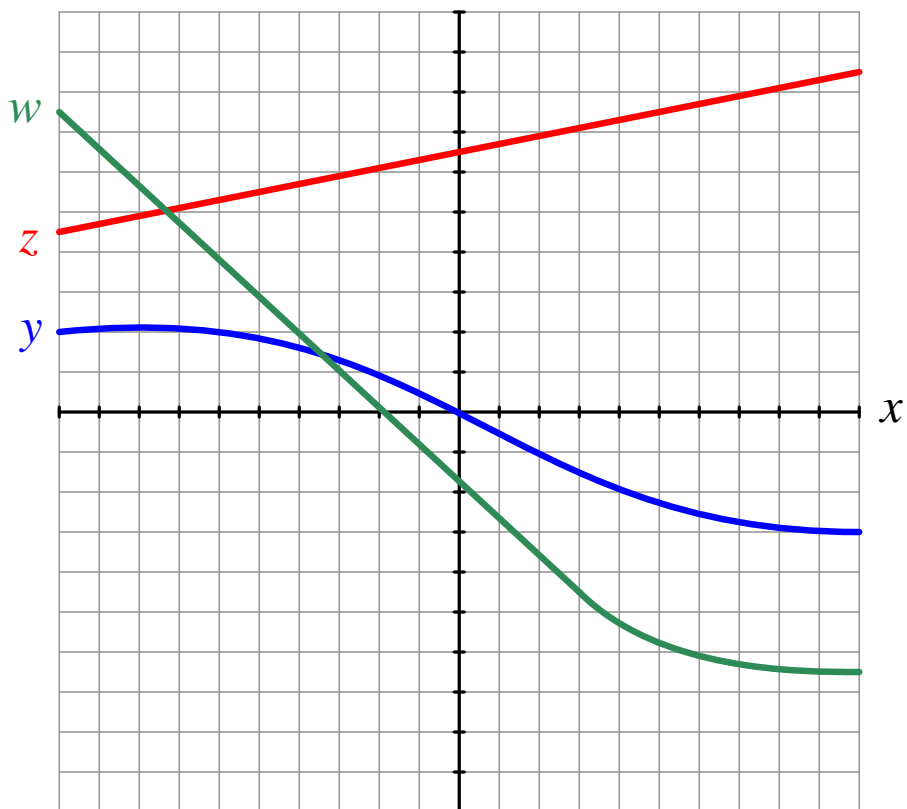


$$\int_{-6}^3 (w - y) dx$$

Also, determine whether the numerical value of this integral is *positive* or *negative*.
[file i01343](#)

Question 5

Shade the area(s) on this graph representing the following integral (assuming each horizontal and vertical division on the graph has an incremental value of 1):

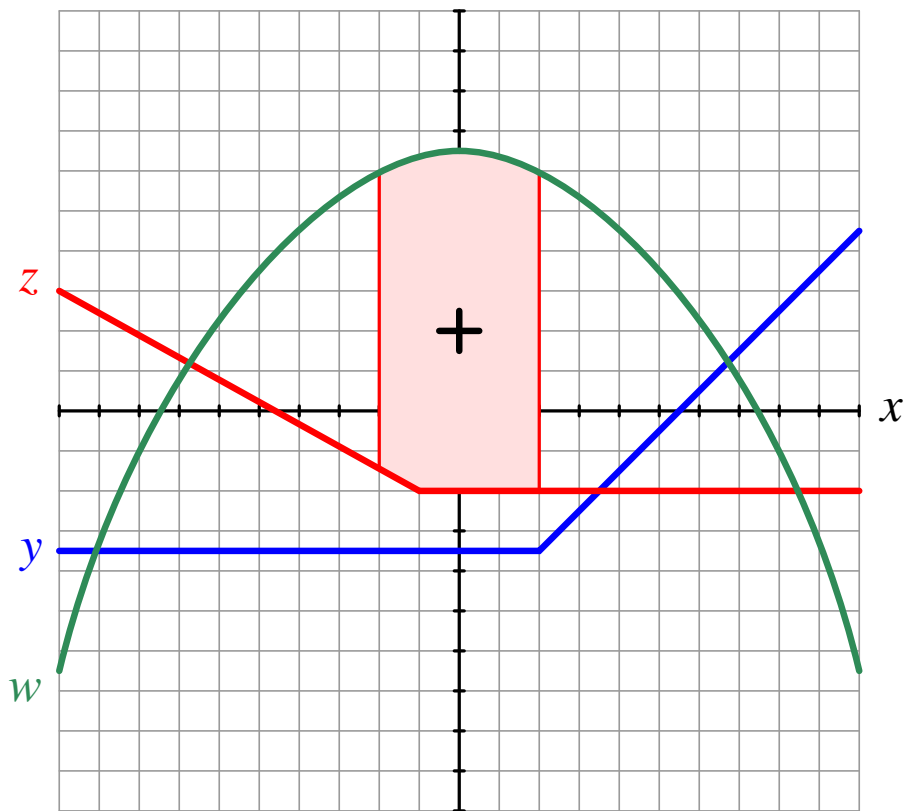


$$\int_{7.5}^4 (w - y) dx$$

Also, determine whether the numerical value of this integral is *positive* or *negative*.
[file i01344](#)

Question 6

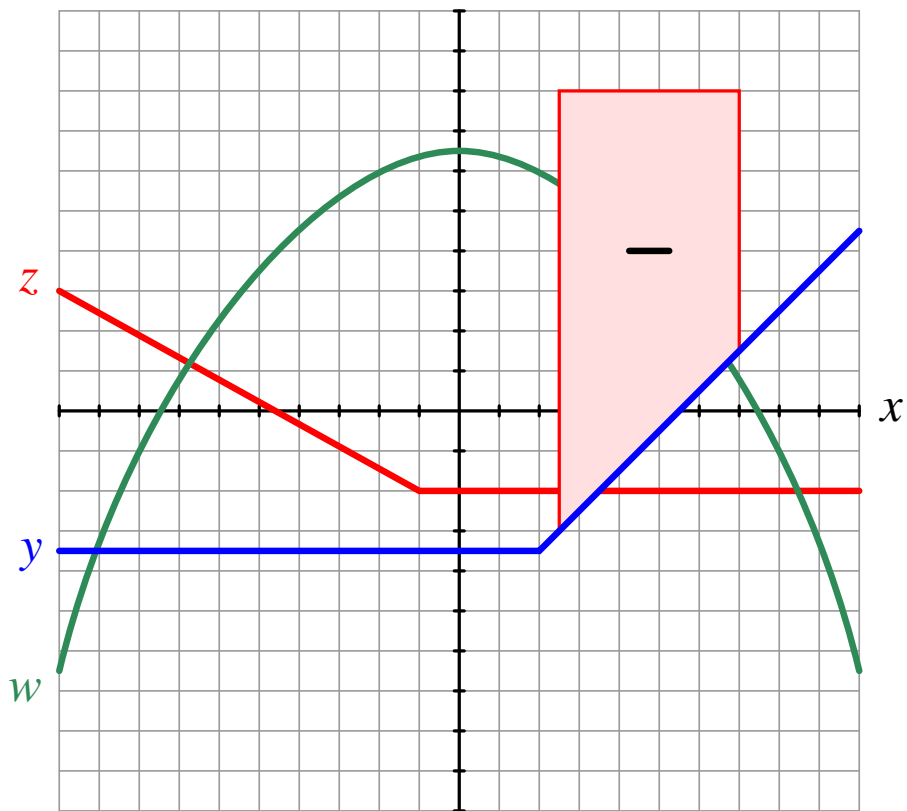
Write the integral expression represented by the shaded area on this graph (assuming each horizontal and vertical division on the graph has an incremental value of 1). The integral for this shaded area has a *positive* value:



file i01346

Question 7

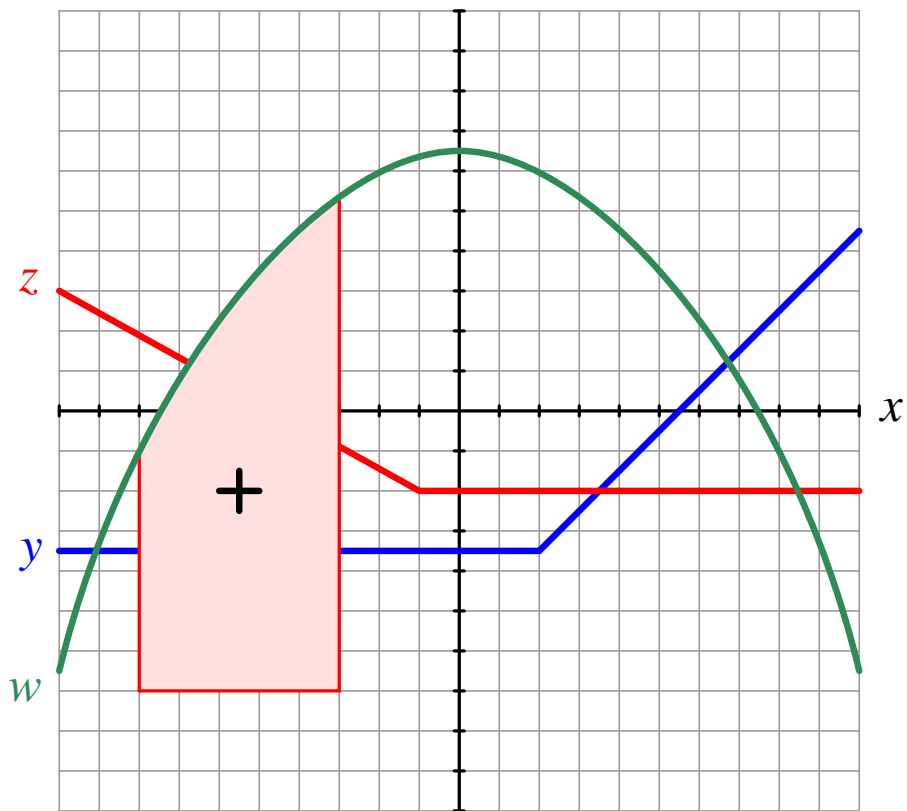
Write the integral expression represented by the shaded area on this graph (assuming each horizontal and vertical division on the graph has an incremental value of 1). The integral for this shaded area has a *negative* value:



file i01905

Question 8

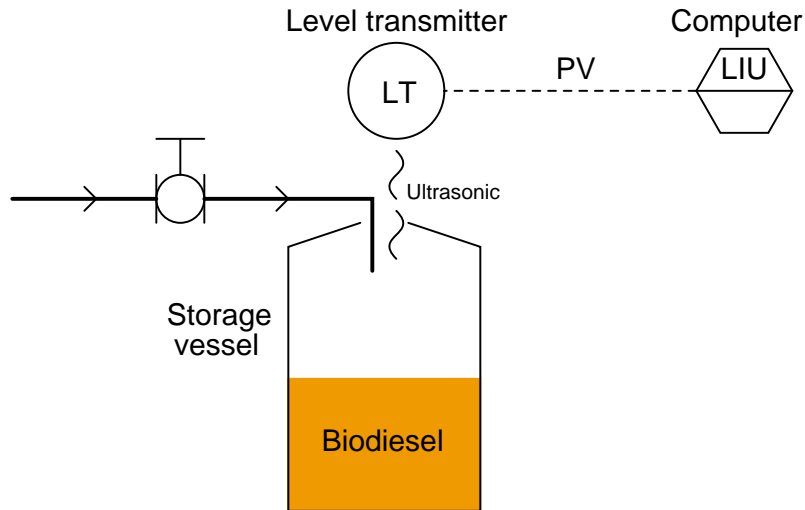
Write the integral expression represented by the shaded area on this graph (assuming each horizontal and vertical division on the graph has an incremental value of 1). The integral for this shaded area has a *positive* value:



file i01906

Question 9

A small-scale biodiesel manufacturing plant records the production of biodiesel fuel by measuring the liquid level in a storage vessel. There is no flow transmitter monitoring flow rate into the vessel, but we can infer flow rate by monitoring vessel level over time.

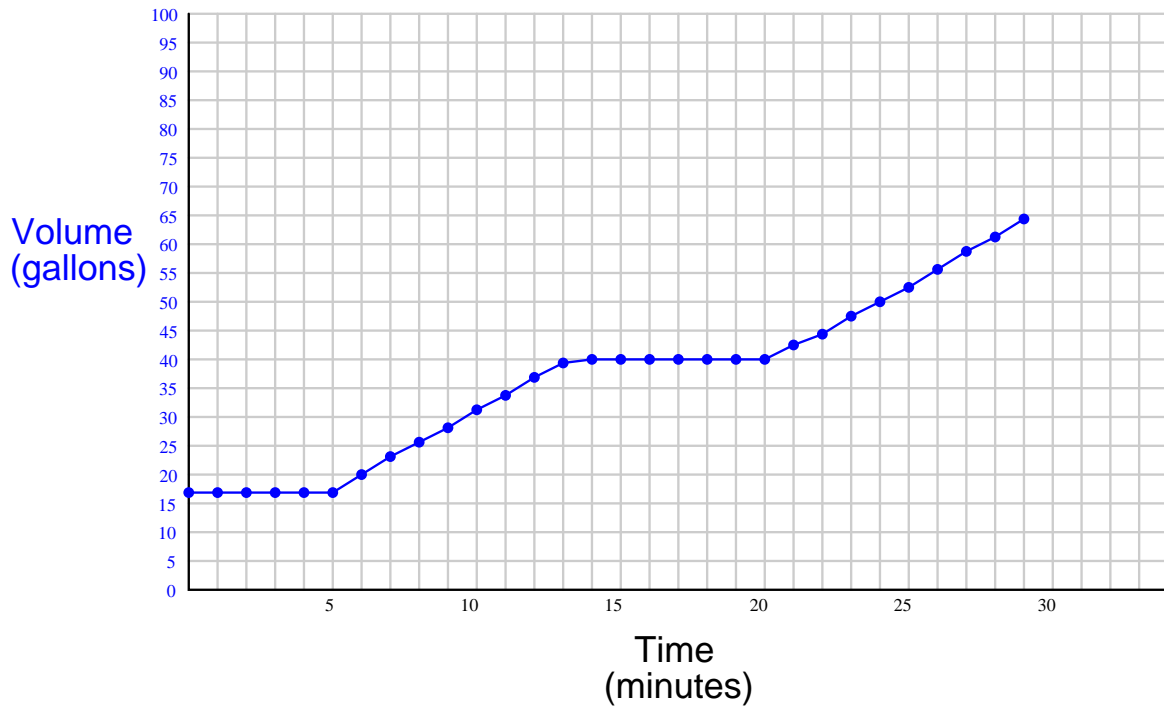


The liquid level measurement signal coming from the ultrasonic level transmitter (LT) is our process variable (PV), and it is sent to a computer to be indicated and processed (LIU). The particular processing done in the computer is calculation of average flow between sample intervals.

Suppose that the computer samples the transmitter's signal once every minute and records these measurements in a data file. Here is an example of that file's contents after one tank-filling batch, shown in a table format:

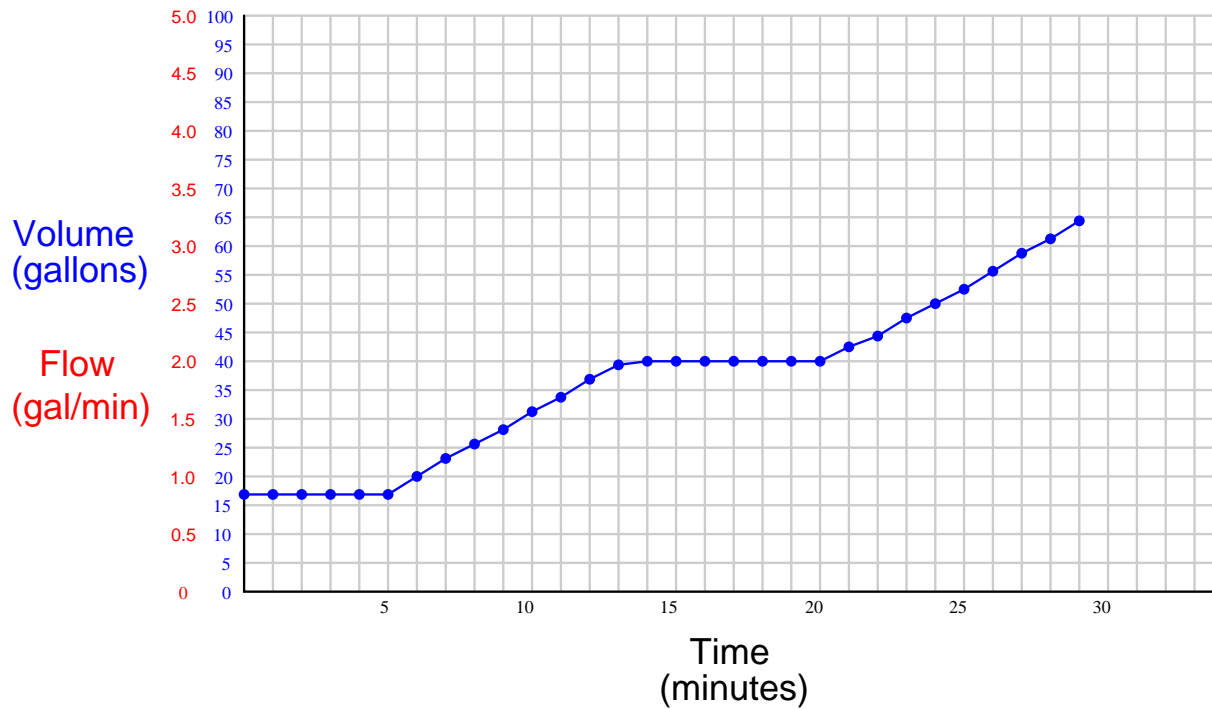
Time (minutes)	Volume (gallons)	Time (minutes)	Volume (gallons)	Time (minutes)	Volume (gallons)
0	17.05	10	31.12	20	40.15
1	17.05	11	33.89	21	42.22
2	17.05	12	36.69	22	44.60
3	17.05	13	39.40	23	47.16
4	17.05	14	40.15	24	50.00
5	17.05	15	40.15	25	52.85
6	20.06	16	40.15	26	55.76
7	23.01	17	40.15	27	58.64
8	25.44	18	40.15	28	61.53
9	28.23	19	40.15	29	64.31

Plotted over time on a graph, this data takes on the following form:



Just looking at this graph, what can you determine about the *flow rate* of biodiesel into this vessel? What would account for the “flat” spot in the middle of the graph? What do the minute variations in slope in the other areas of the graph represent, in terms of flow into the vessel?

Now, use the data in the table to calculate and plot the flow rate of biodiesel in gallons per minute into this vessel:



How do these two plots relate to each other? That is, how does the *shape* of one refer to the *shape* of the other, geometrically?

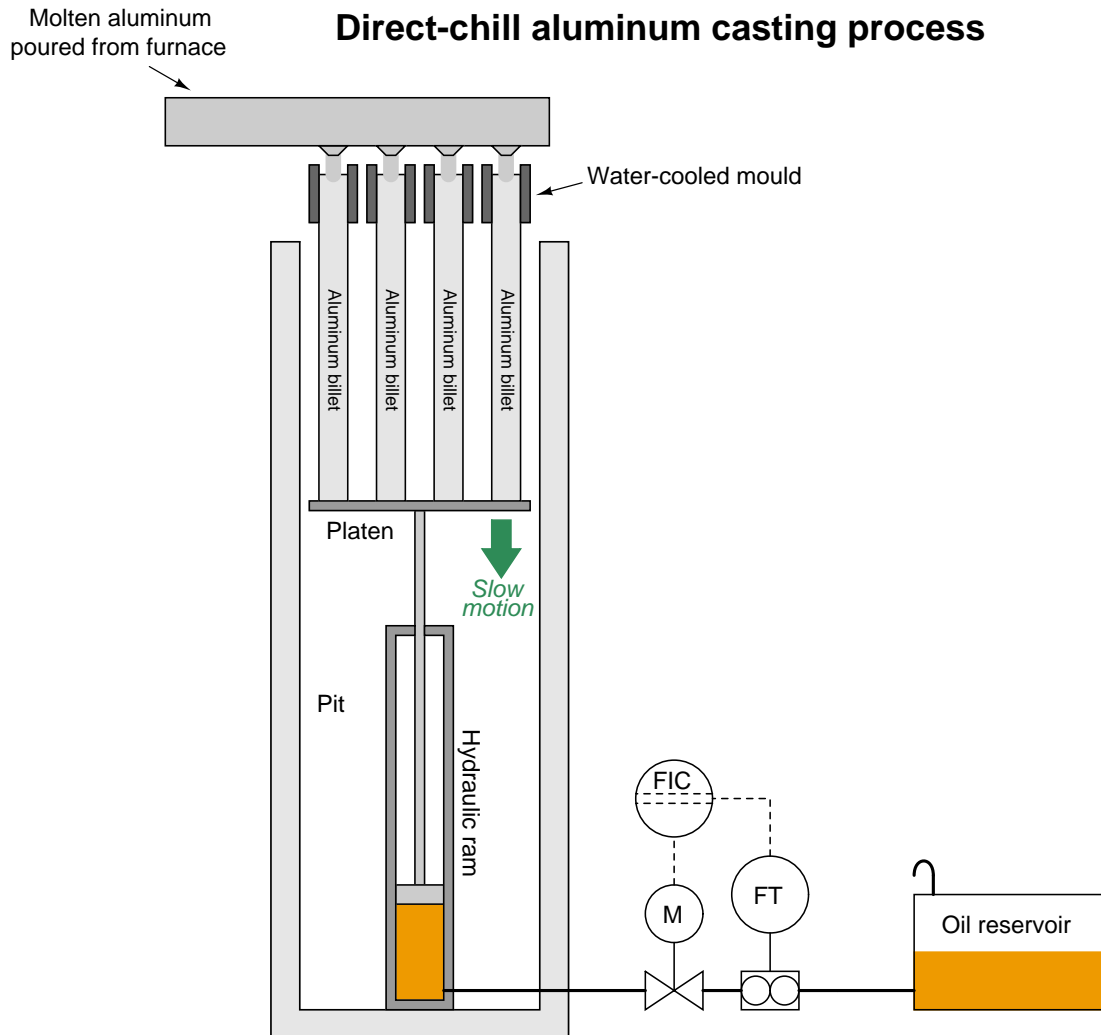
Suggestions for Socratic discussion

- Suppose a technician decided to calculate the flow rate at $t = 4$ minutes by taking the volume at that time (17.05 gallons) and dividing by the time (4 minutes). Explain why this would yield an incorrect value for flow, then explain the correct way to calculate flow rate at that time from the given data.

file i01504

Question 10

Direct-chill aluminum casting is a process by which molten aluminum is cast into long “billets.” The aluminum is poured into a series of water-cooled moulds, solidifying as billets slowly lowered by the downward motion of a hydraulic ram. The motion of this ram is controlled by throttling oil flow out of the bottom, as the ram descends by gravity. Given a known cylinder diameter, the vertical speed of the ram will be a fixed proportion to the volumetric oil flow rate exiting it:



However, if the piston seals within the ram leak, the ram’s descent will not be perfectly proportional to oil flow rate. Consider a case where the normal rate of descent is 11 inches per minute, and the normal “drop” time is 25 minutes to cast billets 275 inches in length. With an empty platen, the descent from leakage is measured by a technician to be 0.3 inches per minute (with the hydraulic control valve shut). With a full platen (aluminum billets of full length), the descent from piston leakage is 1.7 inches per minute (with the hydraulic control valve shut).

Calculate the actual length of the billets given an automatically-controlled flow equivalent to 11 inches per minute, and a “drop” time of 25 minutes. Assume the leakage rate grows linearly over time, from +0.3 inches per minute to +1.7 inches per minute, additional to the controlled flow of 11 inches per minute.

Suggestions for Socratic discussion

- A useful problem-solving technique for calculus-based problems is to sketch a graph of the variables being considered. The *derivative* of a function is the slope of the graph, while the *integral* of a function is the area bound by the graph. Apply this problem-solving technique to the problem at hand in this question.

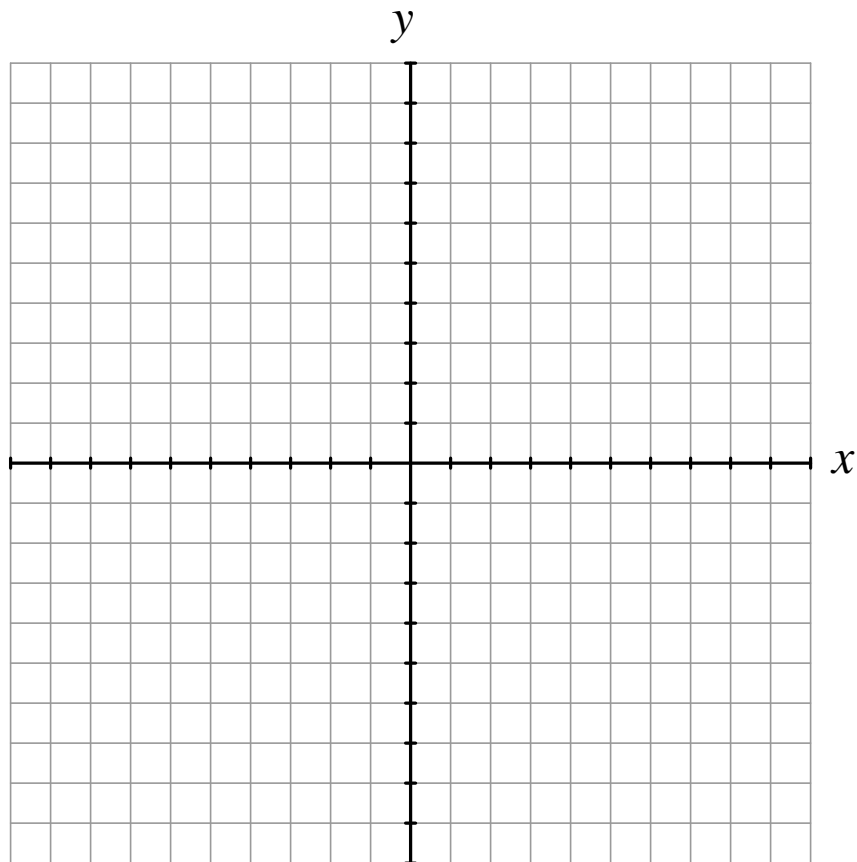
file i02099

Question 11

Complete a table showing x and y values for the following equation, then plot the equation on the coordinate grid provided:

$$y = 3x + 2$$

x	y
-3	
-2	
-1	
0	
1	
2	
3	

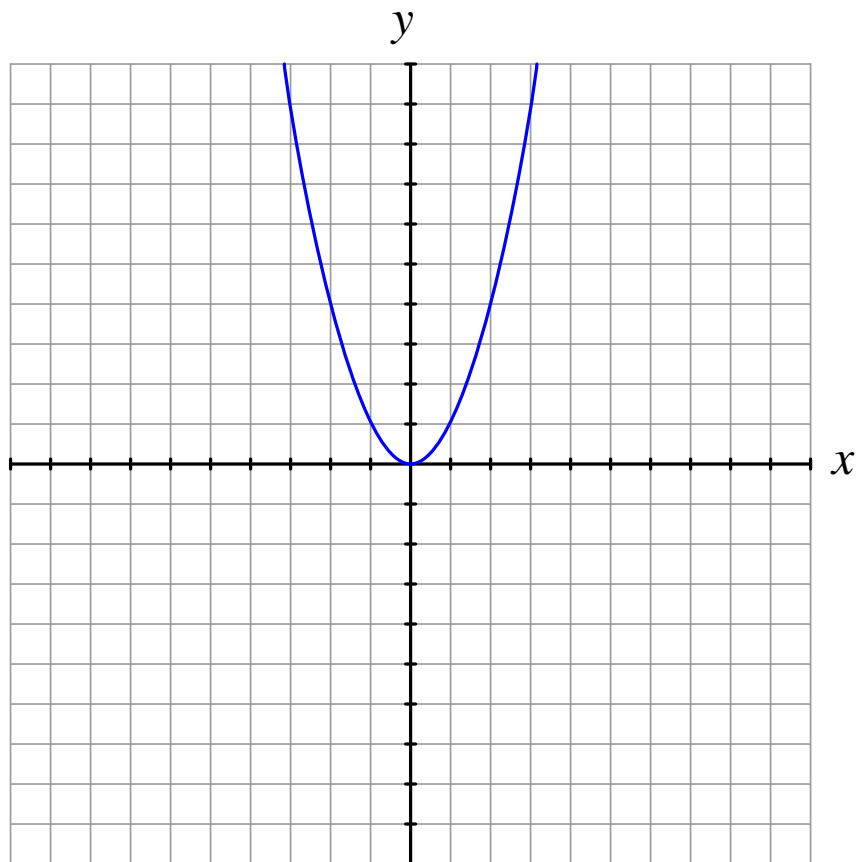


Calculate the slope (at any point) for this equation. Explain both how you may determine the slope by looking at x and y values in the table, and also by examining the written equation.

file i01514

Question 12

Unlike linear functions, the graphs of nonlinear functions do not have constant *slopes*. Take for instance this graph of a “parabola” of the form $y = x^2$:

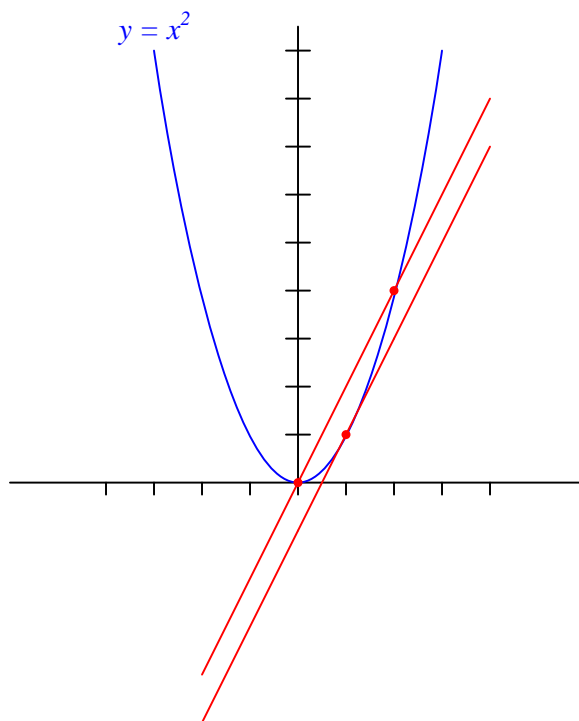


It is difficult (but not impossible) to estimate the slope of a nonlinear function such as this at any given point, just by looking at the graph. If I were to ask you to estimate the slope of this equation's graph at $x = 2$, how would you do it?

[file i01515](#)

Question 13

Two straight lines appear on this graph, along with a *parabola*, defined by the equation $y = x^2$. Dots mark where each of the straight lines intersects the parabola:

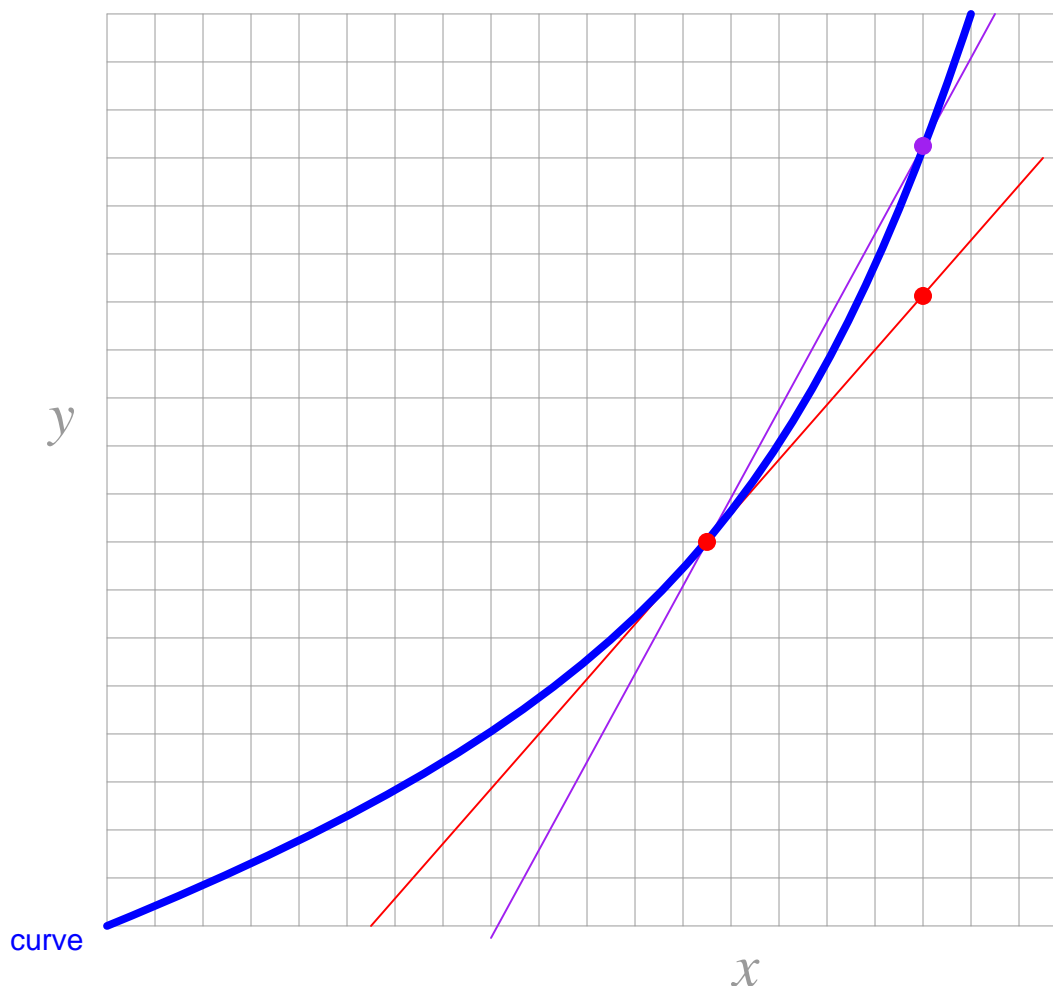


One of these straight lines is a *tangent* line, while the other is a *secant* line. Identify which is which, and then give general definitions for each line type.

[file i01506](#)

Question 14

Shown here is a view of a secant line and a tangent line (both thin) to a curve (thick):



The slope of one of these lines is represented by the ratio of two “increments,” Δy and Δx :

$$\frac{\Delta y}{\Delta x}$$

The slope of the other line is represented by the ratio of two “differentials,” dy and dx , better known as a *derivative*:

$$\frac{dy}{dx}$$

Determine which slope ($\frac{\Delta y}{\Delta x}$ or $\frac{dy}{dx}$) belongs to which type of line (secant or tangent), and label all increments and differentials on the graph.

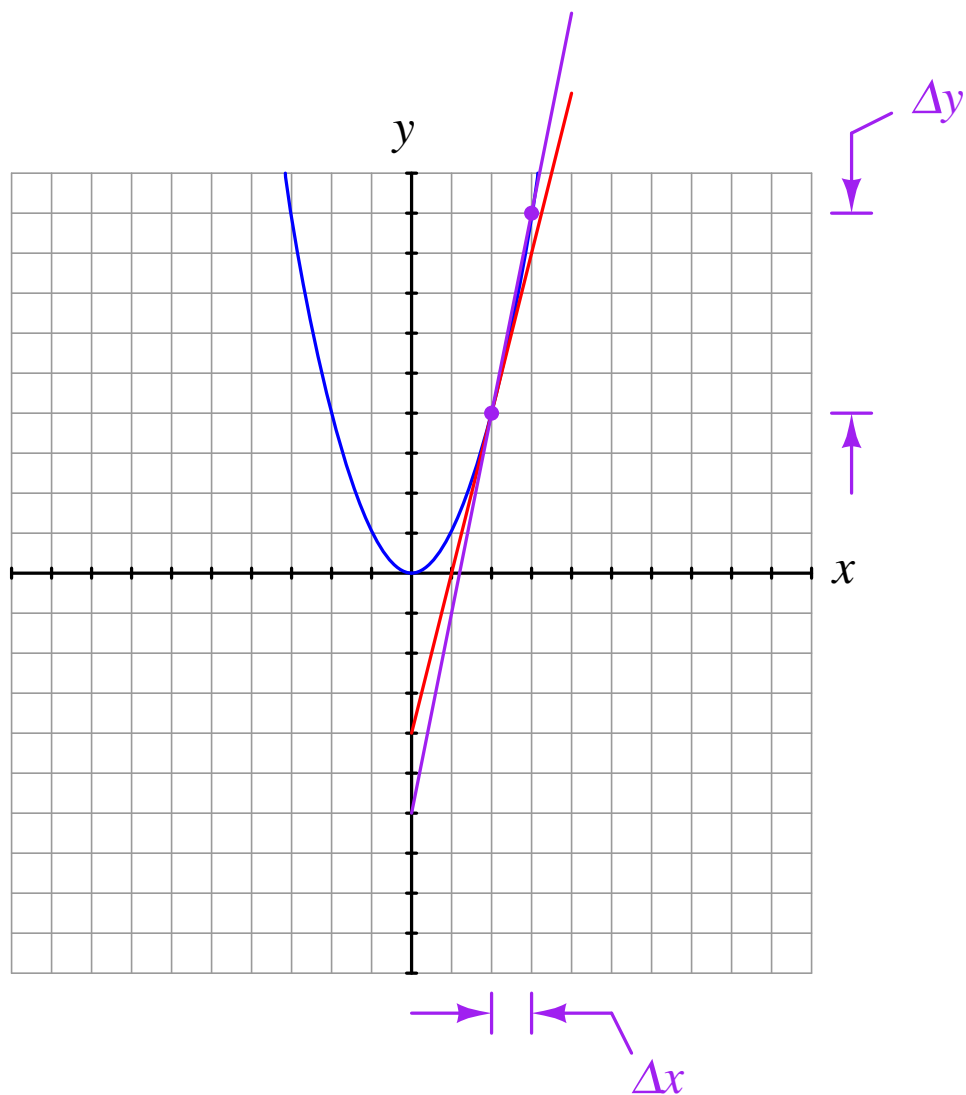
[file i01508](#)

Question 15

Estimate the slope of a parabola ($y = x^2$) at $x = 2$ by calculating the slope of successive secant lines:

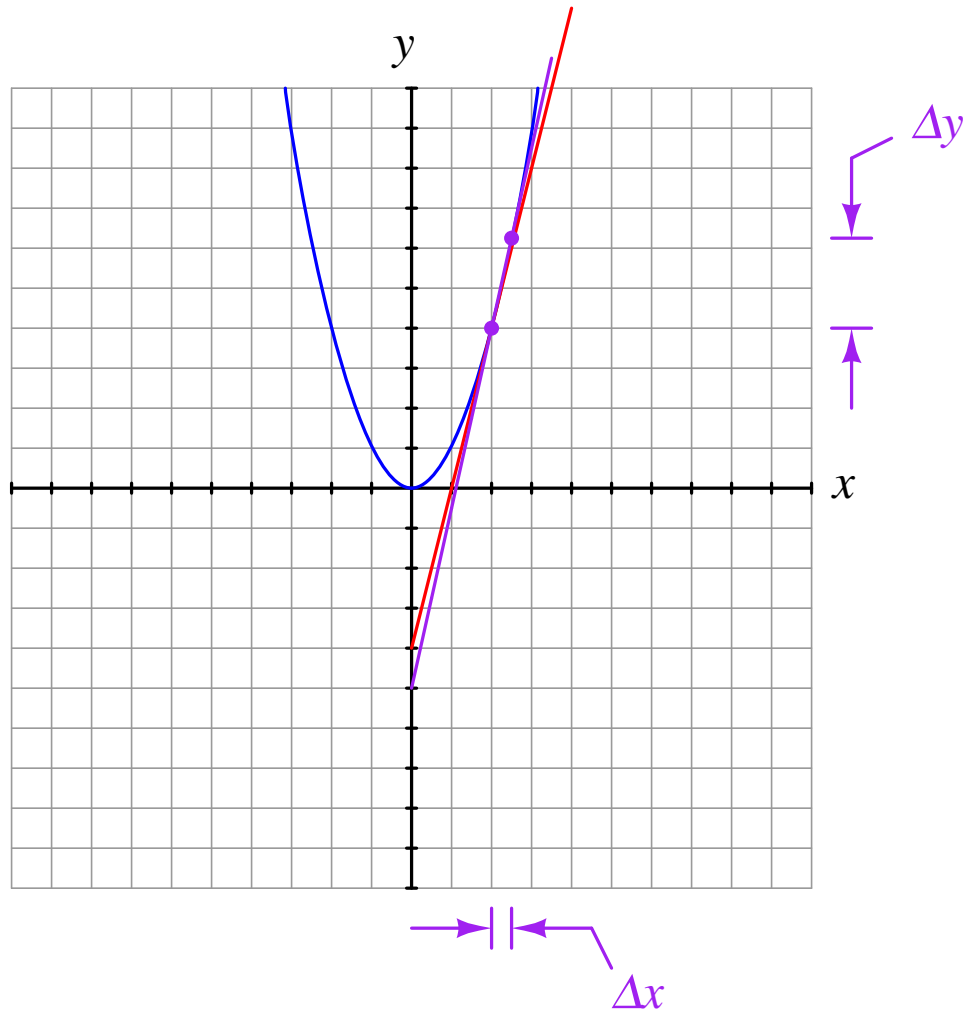
First approximation (points of intersection $x = 2$ and $x = 3$)

Slope = $\frac{\Delta y}{\Delta x} =$ _____



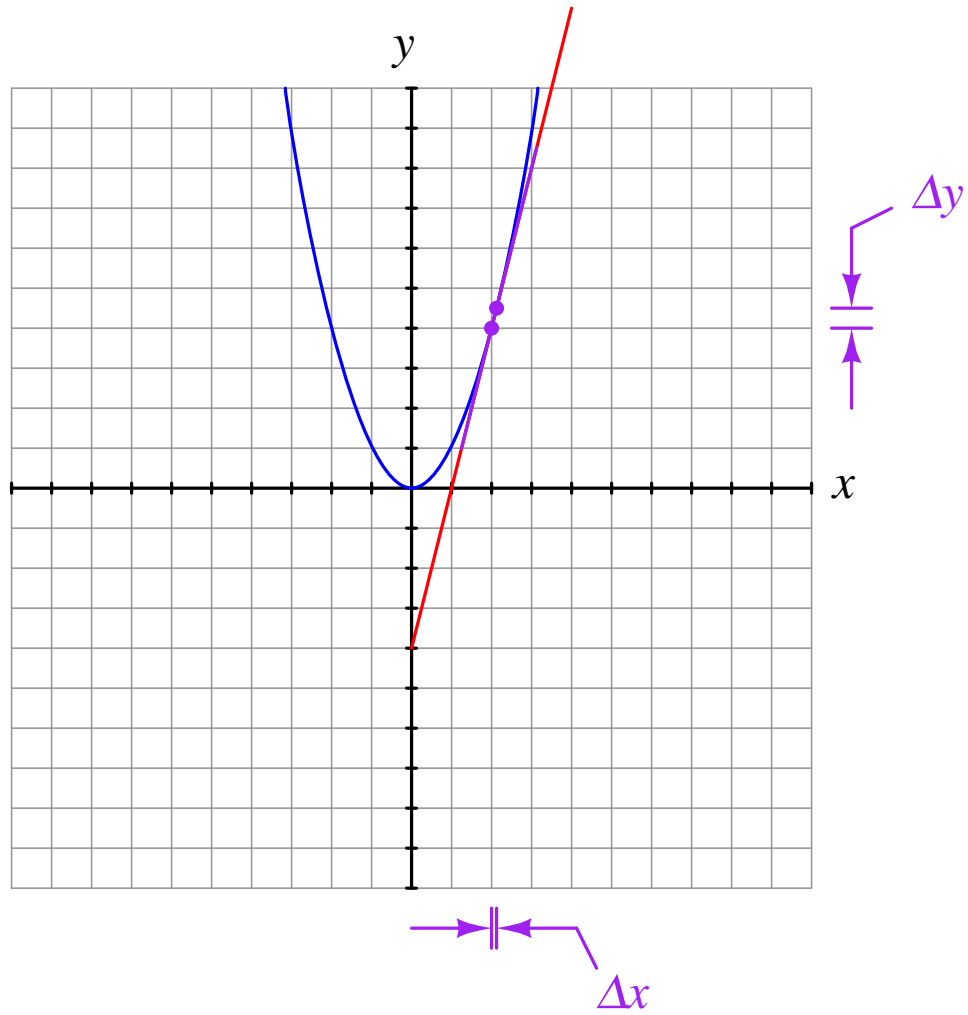
Second approximation (points of intersection $x = 2$ and $x = 2.5$)

Slope = $\frac{\Delta y}{\Delta x} =$ _____



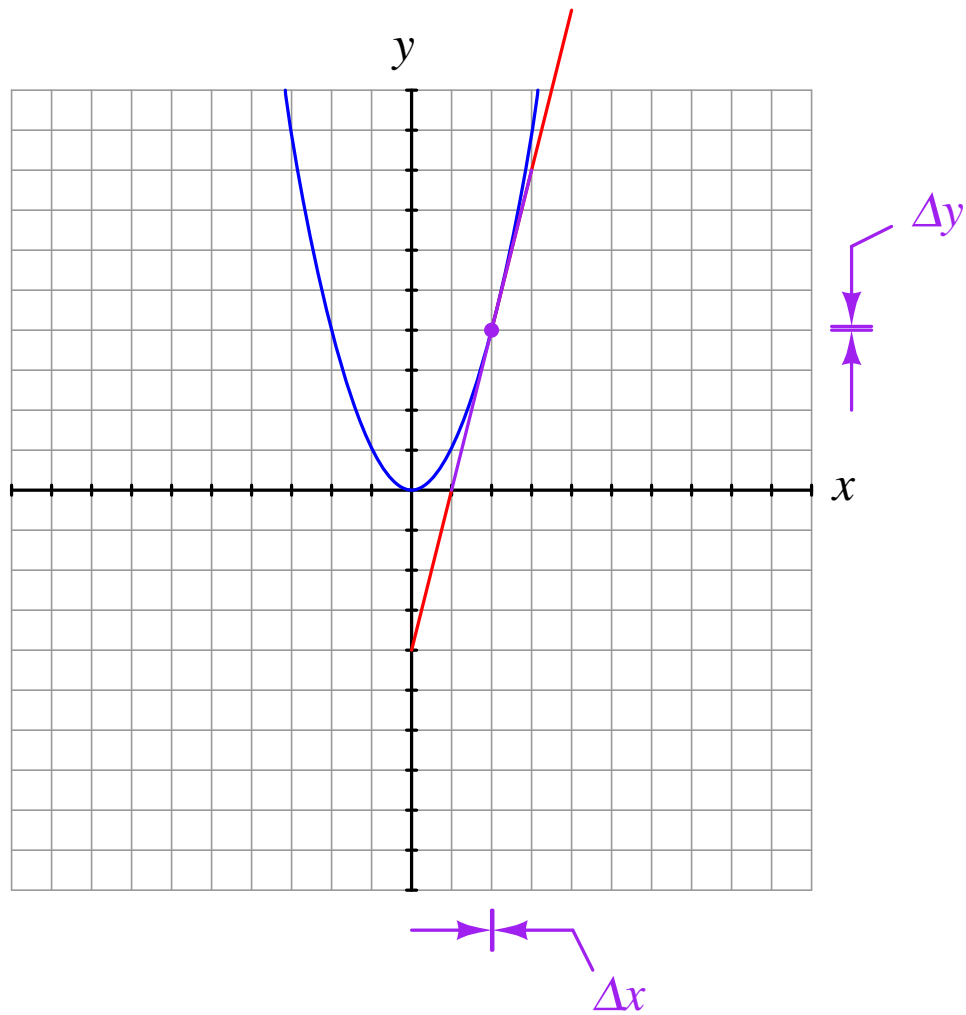
Third approximation (points of intersection $x = 2$ and $x = 2.1$)

Slope = $\frac{\Delta y}{\Delta x} =$ _____



Fourth approximation (points of intersection $x = 2$ and $x = 2.01$)

Slope = $\frac{\Delta y}{\Delta x} =$ _____



Do you see a pattern here? Does the slope of the secant line tend to *approach* a certain value as we make Δx smaller and smaller? How does this secant line slope compare with the slope of the *tangent* line?
[file i01516](#)

Question 16

Estimate the slopes of the tangent lines touching a parabola ($y = x^2$) at $x = 3$, $x = 4$, and $x = 5$ by using secant line approximations:

If $y = x^2$ then,

$$\left. \frac{dy}{dx} \right|_{x=3} =$$

$$\left. \frac{dy}{dx} \right|_{x=4} =$$

$$\left. \frac{dy}{dx} \right|_{x=5} =$$

Do you see a mathematical relationship between the value of x and the slope of the line tangent to that point on the parabola? Can you express that relationship as a function of x ?

$$\frac{dy}{dx} = ???$$

file i01517

Question 17

Estimate the slopes of the tangent lines touching a nonlinear function ($y = x^3$) at $x = 1$, $x = 2$, $x = 3$, $x = 4$, and $x = 5$ by using secant line approximations:

If $y = x^3$ then,

$$\left. \frac{dy}{dx} \right|_{x=1} =$$

$$\left. \frac{dy}{dx} \right|_{x=2} =$$

$$\left. \frac{dy}{dx} \right|_{x=3} =$$

$$\left. \frac{dy}{dx} \right|_{x=4} =$$

$$\left. \frac{dy}{dx} \right|_{x=5} =$$

Do you see a mathematical relationship between the value of x and the slope of the line tangent to that point on the cubic function? Can you express that relationship as a function of x ?

$$\frac{dy}{dx} = ???$$

file i01518

Question 18

The “Power Rule” in differential calculus tells us how to find a formula for the *derivative* of a function consisting of an integer power of the independent variable:

$$\text{If } y = x^n \text{ then } \frac{dy}{dx} = nx^{n-1}$$

Using the Power Rule, differentiate the following functions:

$$y = x \qquad \frac{dy}{dx} =$$

$$y = x^2 \qquad \frac{dy}{dx} =$$

$$y = x^3 \qquad \frac{dy}{dx} =$$

$$y = x^4 \qquad \frac{dy}{dx} =$$

$$y = x^5 \qquad \frac{dy}{dx} =$$

$$y = x^{-1} \qquad \frac{dy}{dx} =$$

$$y = \sqrt{x} \qquad \frac{dy}{dx} =$$

$$y = 11 \qquad \frac{dy}{dx} =$$

file i01519

Question 19

The “Constant Multiple Rule” in differential calculus tells us that the derivative of a function multiplied by a constant will be that constant multiplied by the derivative of the function:

$$\text{If } y = af(x) \text{ then } \frac{dy}{dx} = a \left[\frac{d}{dx} f(x) \right]$$

The “Sum Rule” in differential calculus tells us that the derivative of a sum will be the sum of the derivatives:

$$\text{If } y = f(x) + g(x) \text{ then } \frac{dy}{dx} = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

Both of these rules (plus the “Power Rule”) are useful when we must differentiate a polynomial function such as this:

$$y = 3x^2 + 5x \qquad \frac{dy}{dx} = 6x + 5$$

Apply these rules of differentiation to the following functions:

$$y = 4x^3 + 17x^2 + 2x \qquad \frac{dy}{dx} =$$

$$y = x^5 + 4x^3 + 6x^2 \qquad \frac{dy}{dx} =$$

$$y = -x^2 + 3x + 10 \qquad \frac{dy}{dx} =$$

$$y = 3x^2 - 8x - 1 \qquad \frac{dy}{dx} =$$

$$y = -5x^3 + 10x^2 - 6x \qquad \frac{dy}{dx} =$$

file i01520

Question 20

In physics, a very common formula relating position (x), velocity (v) and constant acceleration (a) is as follows:

$$x = x_o + v_o t + \frac{1}{2} a t^2$$

Where,

x = Final position

x_o = Original position

v_o = Original velocity

a = Acceleration

t = Time

Differentiate this equation with respect to time, showing your work at each step. Also, explain the meaning of the derived equation (what does $\frac{dx}{dt}$ represent in physics?).

[file i01522](#)

Question 21

Capacitors store energy in the form of an electric field. We may calculate the energy stored in a capacitance by integrating the product of capacitor voltage and capacitor current ($P = IV$) over time, since we know that power is the rate at which work (W) is done, and the amount of work done to a capacitor taking it from zero voltage to some non-zero amount of voltage constitutes energy stored (U):

$$P = \frac{dW}{dt}$$

$$dW = P dt$$

$$U = W = \int P dt$$

Find a way to substitute capacitance (C) and voltage (V) into the integrand so you may integrate to find an equation describing the amount of energy stored in a capacitor for any given capacitance and voltage values.

[file i02464](#)

Question 22

Inductors store energy in the form of a magnetic field. We may calculate the energy stored in an inductance by integrating the product of inductor voltage and inductor current ($P = IV$) over time, since we know that power is the rate at which work (W) is done, and the amount of work done to an inductor taking it from zero current to some non-zero amount of current constitutes energy stored (U):

$$P = \frac{dW}{dt}$$

$$dW = P dt$$

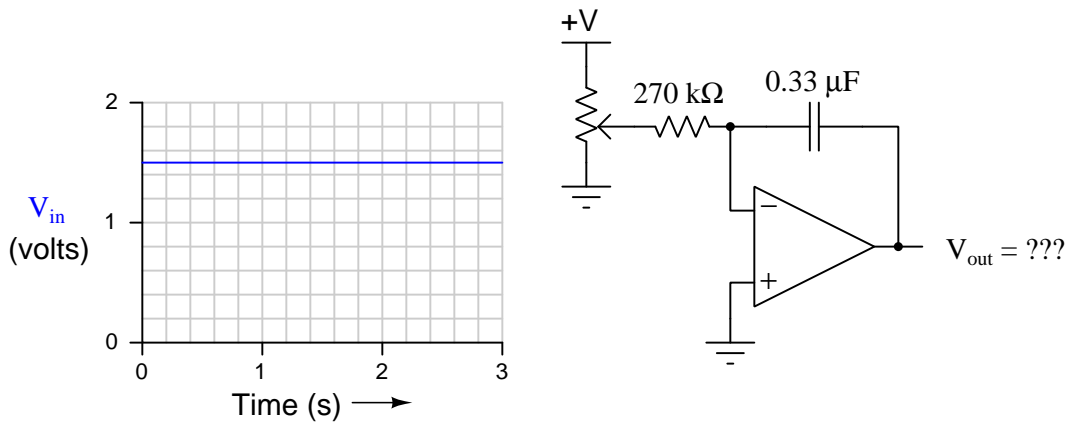
$$U = W = \int P dt$$

Find a way to substitute inductance (L) and current (I) into the integrand so you may integrate to find an equation describing the amount of energy stored in an inductor for any given inductance and current values.

[file i02465](#)

Question 23

Calculate the output voltage of this opamp circuit assuming a constant input voltage of +1.5 volts applied for 0.5 seconds. Assume the capacitor begins in a state of zero charge ($V_0 = 0$ volts):

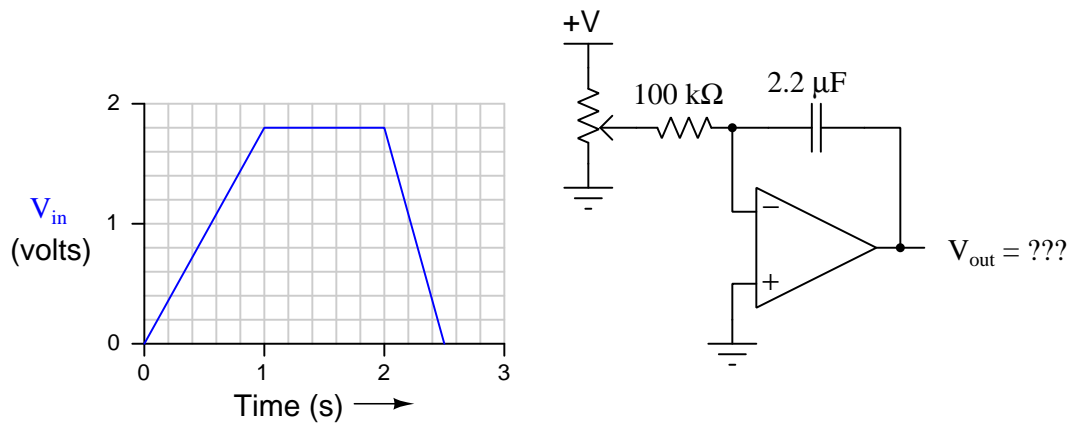


Also, calculate the “time constant” of this integrator circuit (i.e. the factor relating output voltage rate-of-change to input voltage).

[file i01026](#)

Question 24

Calculate the output voltage of this opamp circuit given the input voltage profile shown in the graph. Assume the capacitor begins with a charge (V_0) of +8 volts:

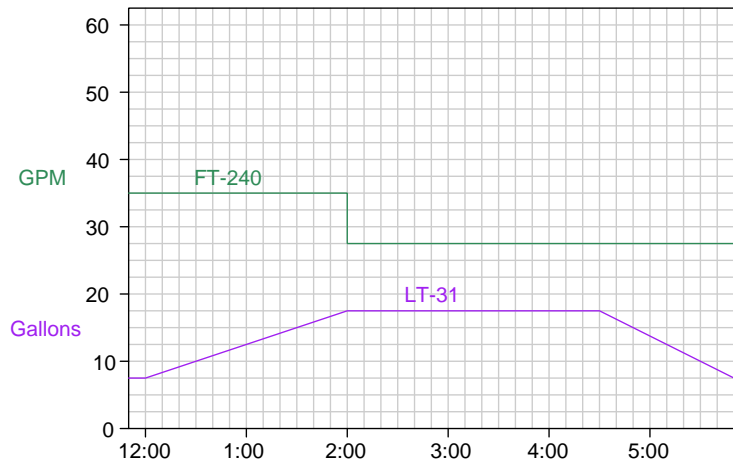
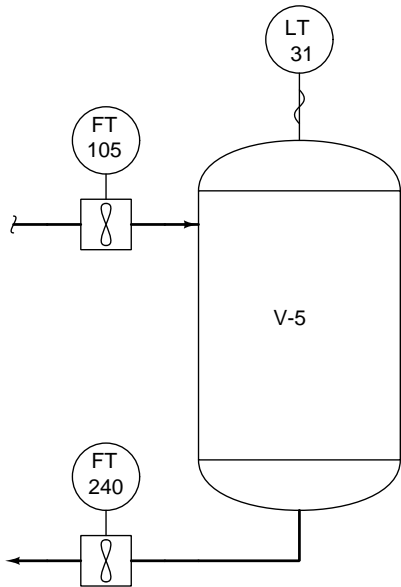


Also, calculate the “time constant” of this integrator circuit (i.e. the factor relating output voltage rate-of-change to input voltage).

[file i01027](#)

Question 25

Calculate the rate of liquid flow coming into process vessel V-5 at 1:30 PM, and also at 3:45 PM, based on the information shown here:



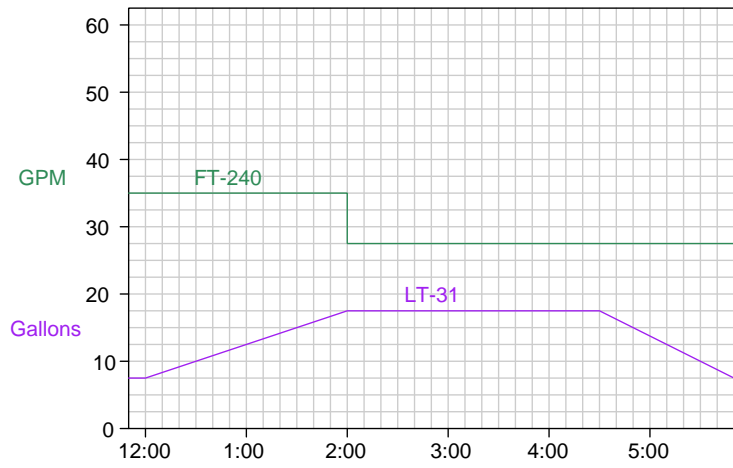
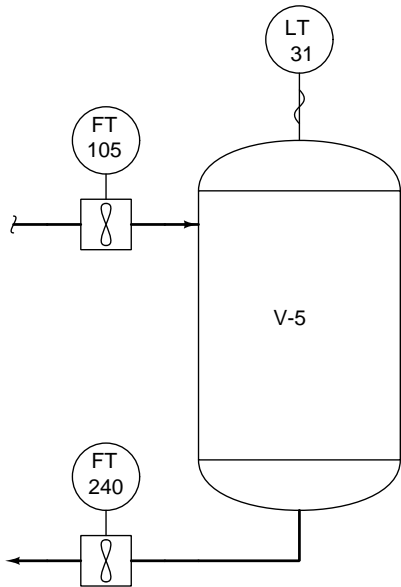
Q_{in} @ 1:30 PM = _____ GPM

Q_{in} @ 3:45 PM = _____ GPM

[file i02890](#)

Question 26

Calculate the volume of liquid discharged from this vessel between 1:00 PM and 4:00 PM based on the information shown here:

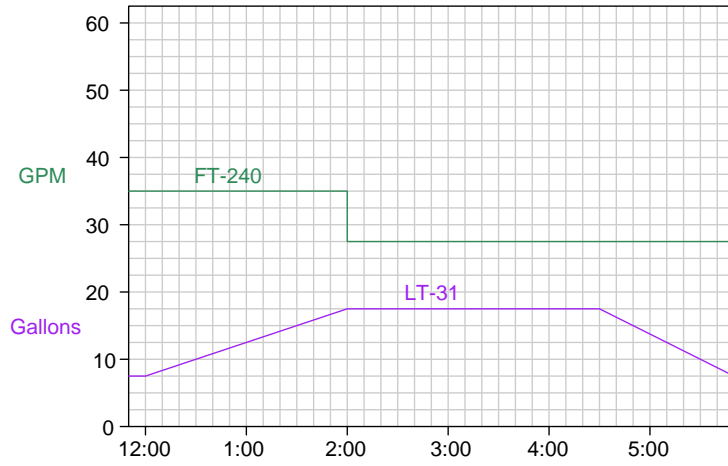
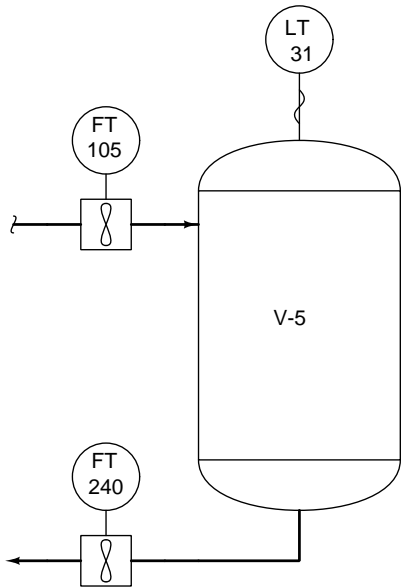


$V_{discharged}$ between 1:00 PM and 4:00 PM = _____ gallons

[file i02889](#)

Question 27

Calculate the amount of liquid lost from the vessel between 4:30 PM and 5:30 PM based on the information shown here:

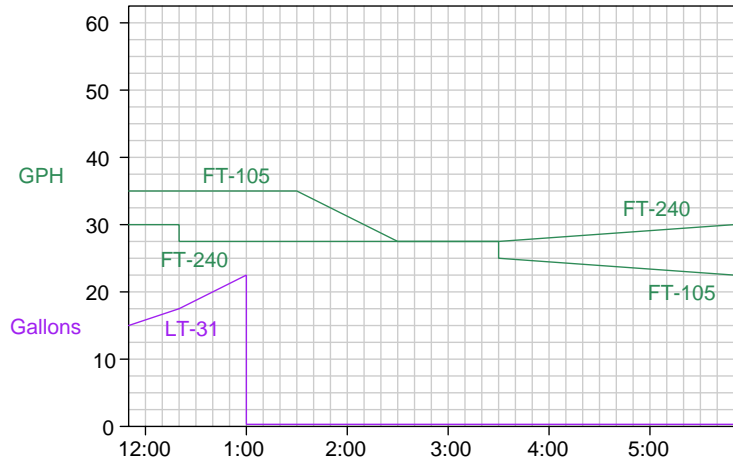
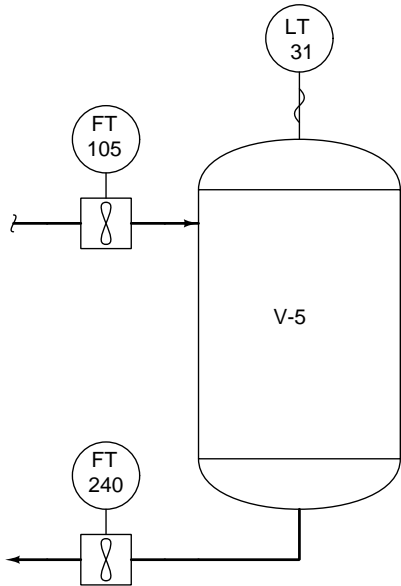


V_{lost} between 4:30 PM and 5:30 PM = _____ gallons

[file i02888](#)

Question 28

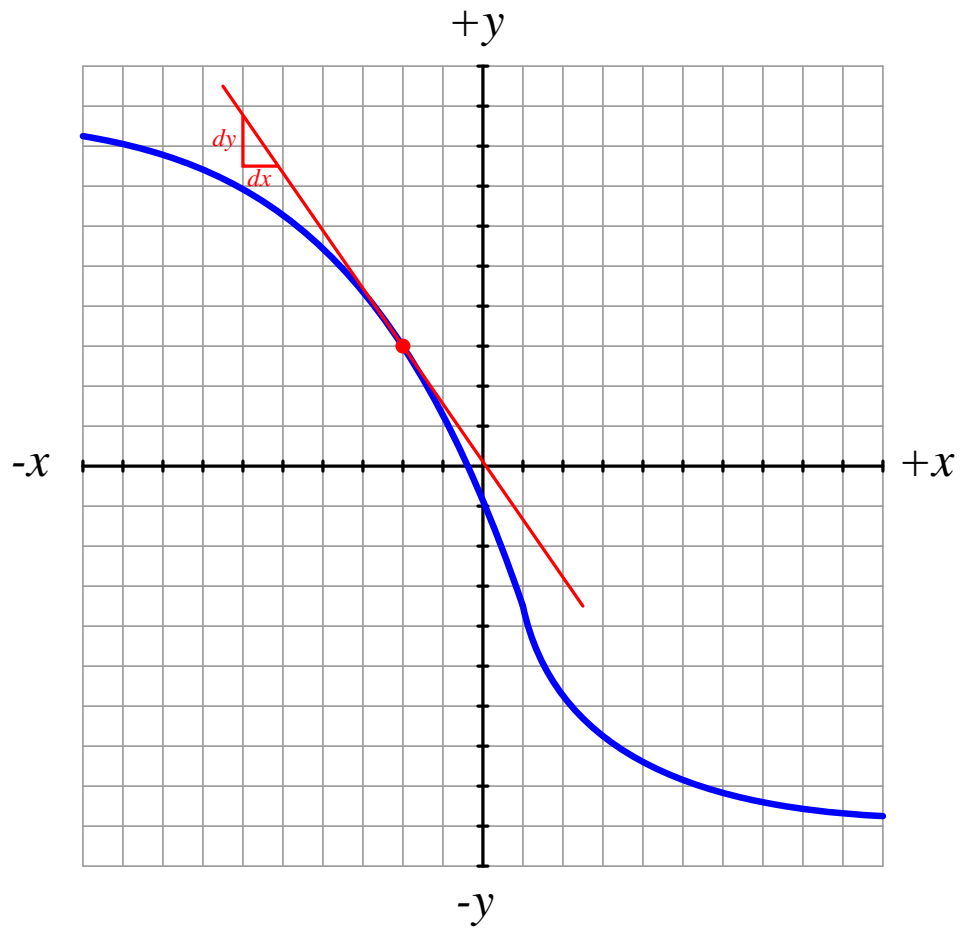
The level transmitter (LT-31) on vessel V-5 failed at 1:00 PM today, its signal going all the way to zero even though there was still liquid inside the vessel. You are asked to calculate the amount of liquid in the vessel at 3:00 PM based on the flow trends shown (in units of gallons per *hour*):



V @ 3:00 PM = _____ gallons

file i02886

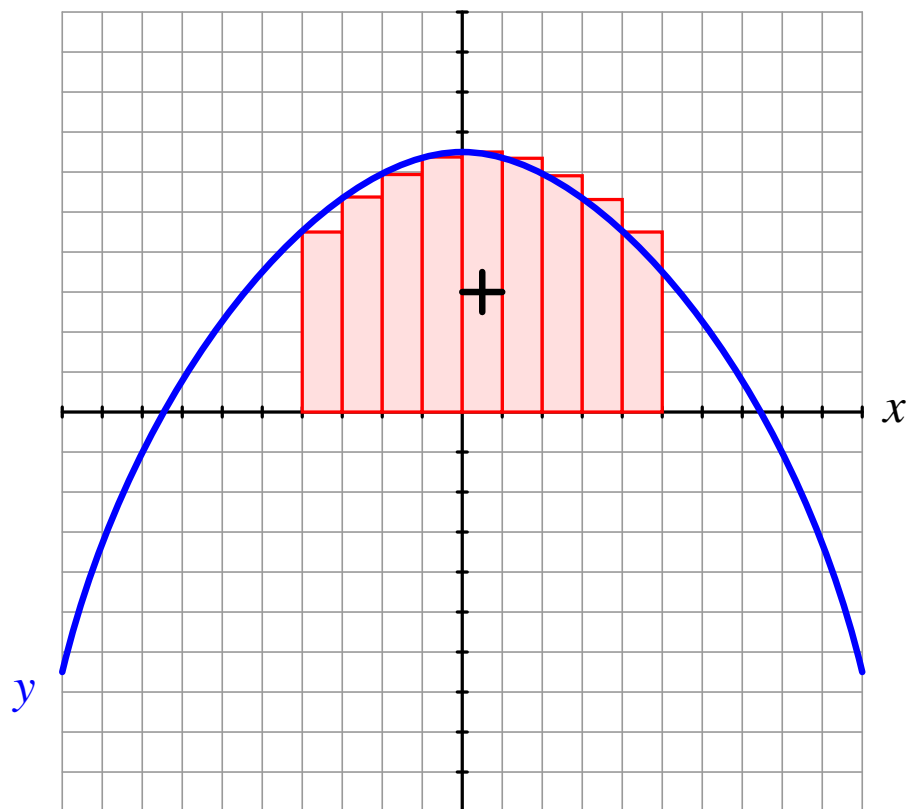
Answer 1



$\frac{dy}{dx}$ has a *negative* value at the red dot because the differential dy is negative while the differential dx is positive.

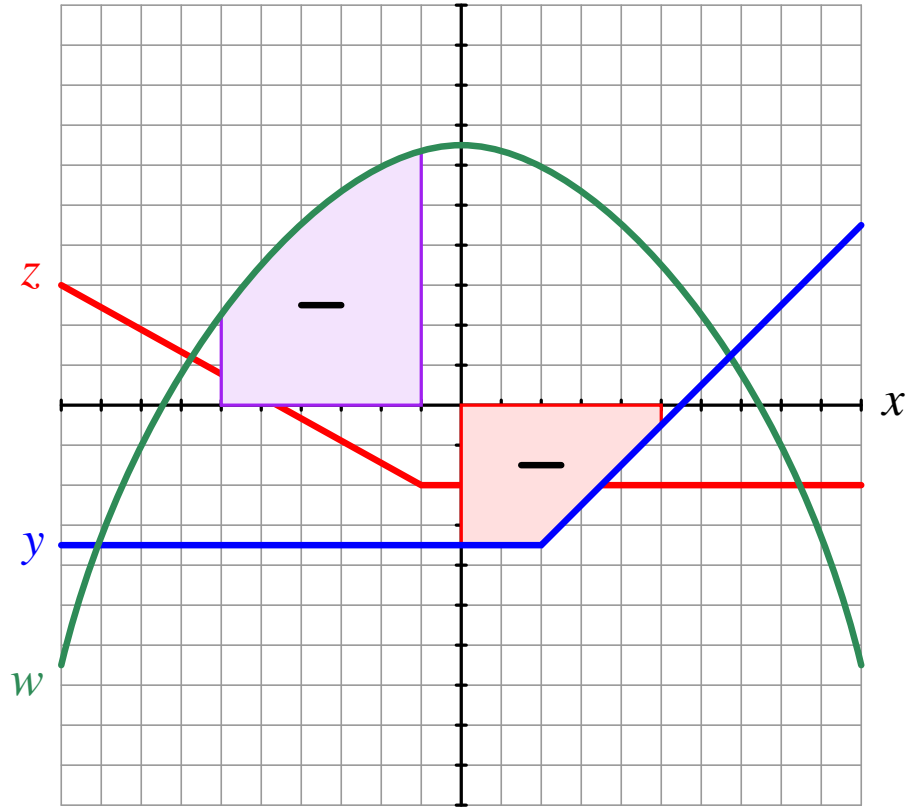
Answer 2

This Riemann sum has a *positive* value, because each rectangular area has a positive value (each one having a positive y value multiplied by positive Δx value):



Answer 3

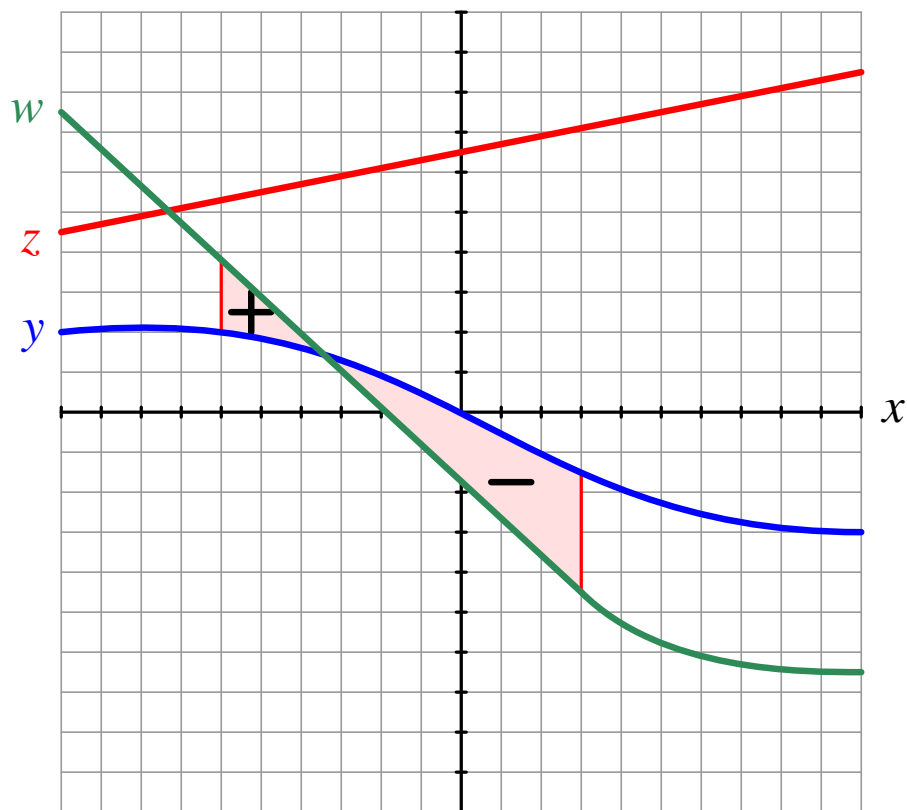
Both integrals have *negative* values:



The first integral (shaded *red*) has positive dx increments but negative y values. The second integral (shaded *violet*) has negative dx increments and positive w values.

Answer 4

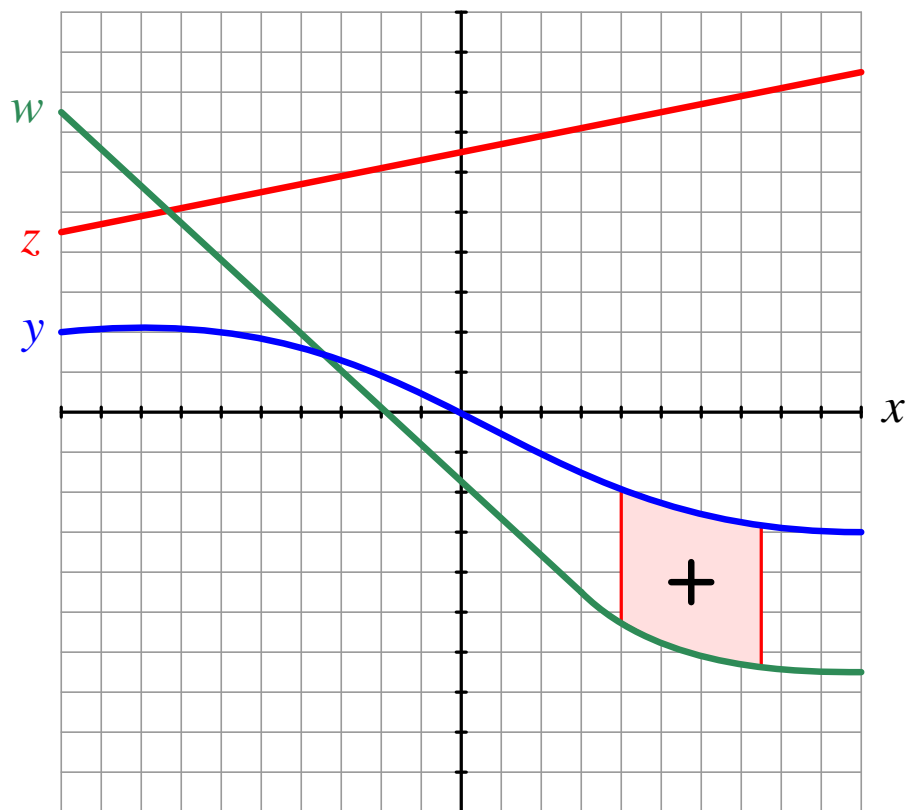
This integral has a *negative* value:



From $x = -6$ to approximately $x = -3.5$, the integral accumulates a positive value because it has positive dx increments and w is larger than y . From approximately $x = -3.5$ to $x = 3$ the integral accumulates a negative value because y is larger than w (a negative integrand) while dx increments still remain positive. Overall, the negative area is greater than the positive area, and so the integral from $x = -6$ to $x = 3$ has a net negative value.

Answer 5

This integral has a *positive* value:



Answer 6

There are two different ways to write this integral:

$$\int_{-2}^2 (w - z) dx$$
$$\int_2^{-2} (z - w) dx$$

Answer 7

There are two different ways to write this integral:

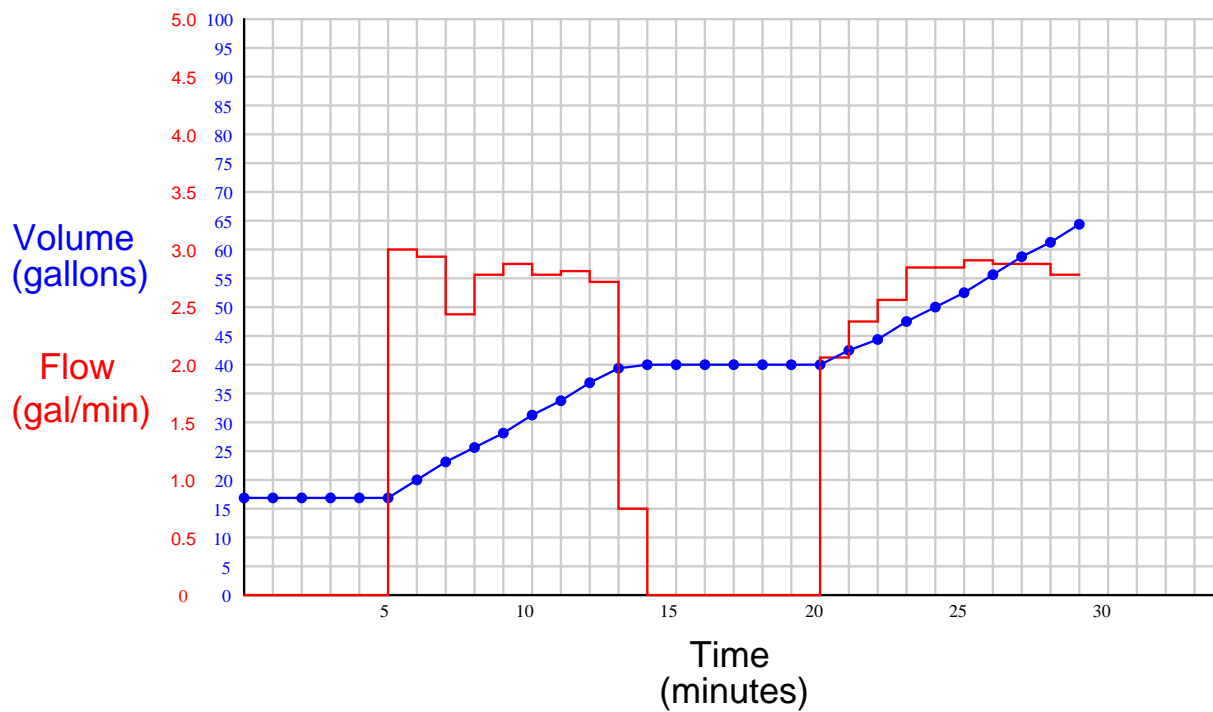
$$\int_7^{2.5} (8 - y) dx$$
$$\int_{2.5}^7 (y - 8) dx$$

Answer 8

There are two different ways to write this integral:

$$\int_{-8}^3 (w + 7) dx$$
$$\int_3^{-8} -(w + 7) dx$$

Answer 9



Answer 10

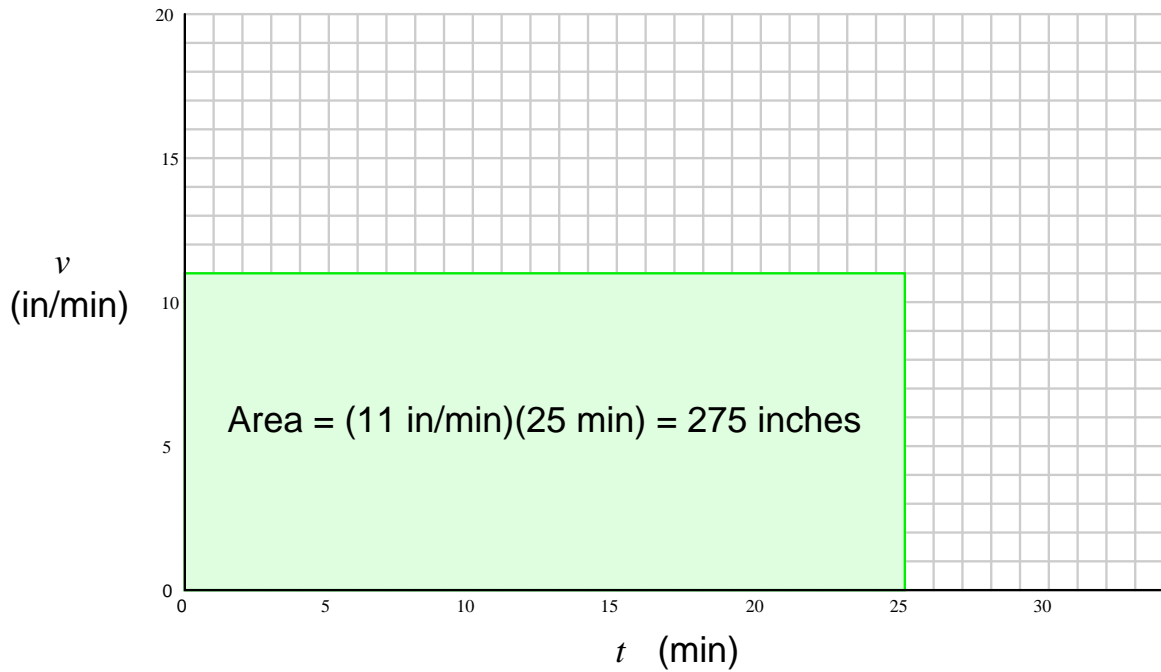
This is an *integration* problem, calculating distance (x) given velocity (v) and time (t). With a constant velocity, distance is the product of velocity and time:

$$x = vt$$

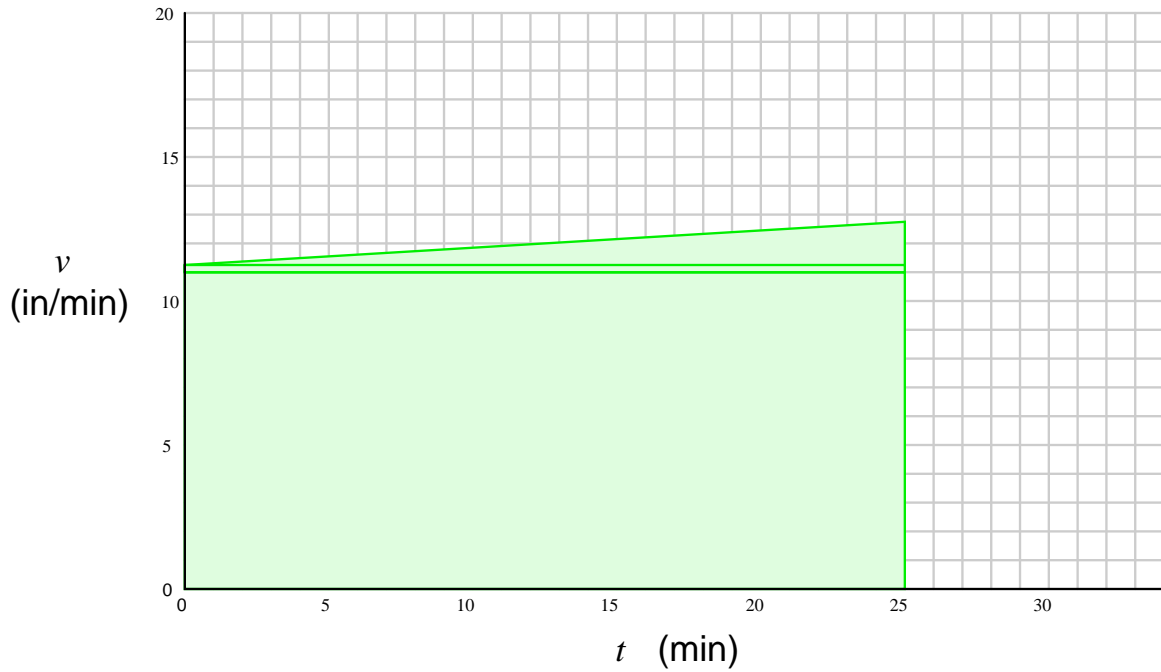
However, in this case we do not have a constant velocity. The velocity of the billets increases over time, due to the increasing platen weight and the piston leak. Distance traveled by the platen is best calculated using the following integral:

$$x = \int v dt$$

With no piston leakage, the integral may be expressed graphically as the area enclosed by a line 11 inches per minute high, and 25 minutes wide:



With leakage (starting at 0.3 inches per minute and progressing to 1.7 inches per minute), the integral area takes on a trapezoidal shape:



The total area of this shape is 300 inches. Broken up into three distinct geometric shapes (two rectangles and one triangle):

$$x = \int_{0 \text{ min}}^{25 \text{ min}} (11 \text{ in/min}) dt = 275 \text{ inches}$$

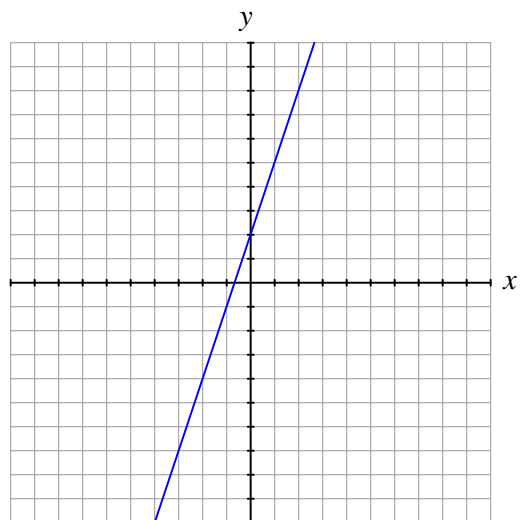
$$x = \int_{0 \text{ min}}^{25 \text{ min}} (0.3 \text{ in/min}) dt = 7.5 \text{ inches}$$

$$x = \int_{0 \text{ min}}^{25 \text{ min}} (0 \text{ to } 1.4 \text{ in/min}) dt = 17.5 \text{ inches}$$

Total distance traveled = 275 inches + 7.5 inches + 17.5 inches = 300 inches.

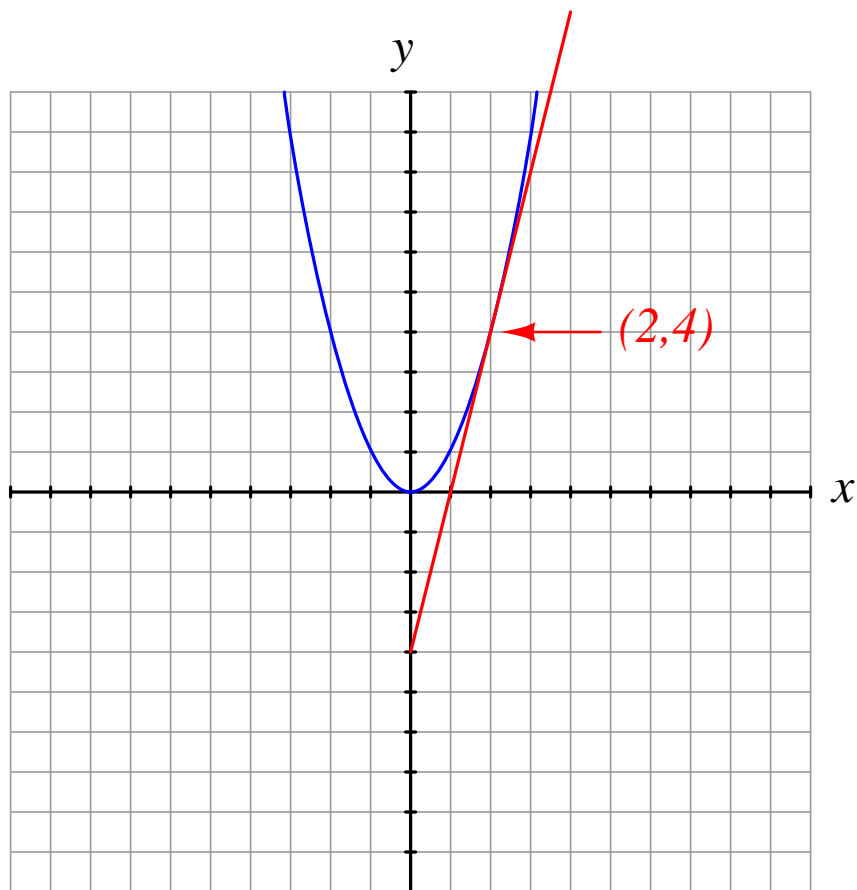
Answer 11

Slope = 3



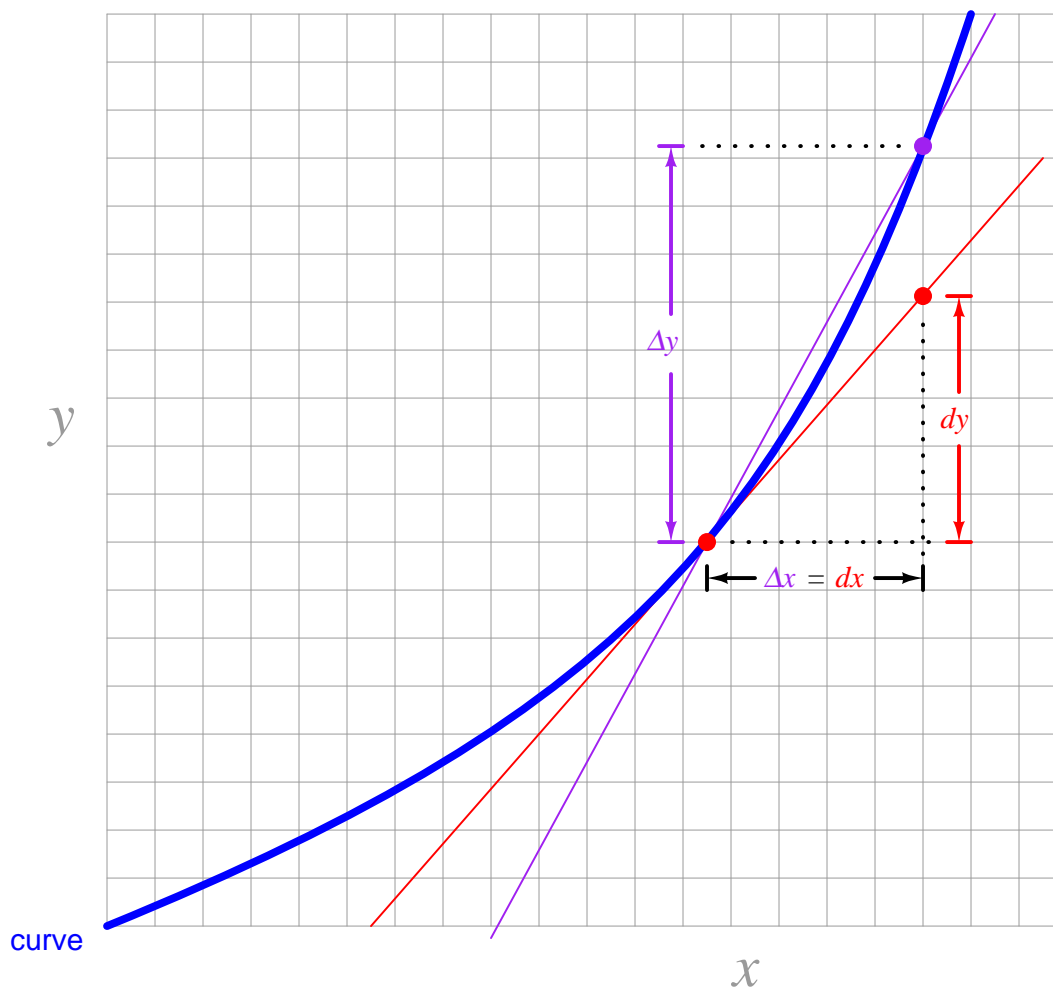
Answer 12

One way is to draw a line that just grazes the curve of the parabola at the coordinate $(2,4)$ and estimate its slope:



Answer 13

A *secant line* intersects a curve at two points, approximating the slope of that curve at some point in between the two intersections. A *tangent line*, on the other hand, intersects a curve at a single point without crossing that curve, the slope of that line equaling the slope of that curve at the exact point of intersection.



$$\frac{\Delta y}{\Delta x} = \text{Slope of secant line}$$

$$\frac{dy}{dx} = \text{Slope of tangent line}$$

Due to the classic definition of the derivative as being the limit of the secant line slope as Δx approaches zero, many people are initially led to believe that dx and dy must necessarily be very small. Not so! Differentials such as dx and dy may assume any non-zero real number value, it's just that they indicate the slope of the tangent line, not the slope of the curve between intervals. For practical applications of large-valued differentials, consult an introductory calculus book on the subject of *linear approximations* using differentials.

Answer 15

First approximation (points of intersection $x = 2$ and $x = 3$)

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \underline{\mathbf{5}}$$

Second approximation (points of intersection $x = 2$ and $x = 2.5$)

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \underline{\mathbf{4.5}}$$

Third approximation (points of intersection $x = 2$ and $x = 2.1$)

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \underline{\mathbf{4.1}}$$

Fourth approximation (points of intersection $x = 2$ and $x = 2.01$)

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \underline{\mathbf{4.01}}$$

Fine approximation (points of intersection $x = 2$ and $x = 2.00001$)

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \underline{\mathbf{4.00001}}$$

The slope of the secant line approaches 4 (which is the exact slope of the tangent line at $x = 2$) as we make Δx smaller and smaller.

Answer 16

$$\left. \frac{dy}{dx} \right|_{x=3} = 6$$

$$\left. \frac{dy}{dx} \right|_{x=4} = 8$$

$$\left. \frac{dy}{dx} \right|_{x=5} = 10$$

$$\frac{dy}{dx} = 2x$$

$$\left. \frac{dy}{dx} \right|_{x=1} = 3$$

$$\left. \frac{dy}{dx} \right|_{x=2} = 12$$

$$\left. \frac{dy}{dx} \right|_{x=3} = 27$$

$$\left. \frac{dy}{dx} \right|_{x=4} = 48$$

$$\left. \frac{dy}{dx} \right|_{x=5} = 75$$

$$\frac{dy}{dx} = 3x^2$$

$$y = x \qquad \frac{dy}{dx} = 1$$

$$y = x^2 \qquad \frac{dy}{dx} = 2x$$

$$y = x^3 \qquad \frac{dy}{dx} = 3x^2$$

$$y = x^4 \qquad \frac{dy}{dx} = 4x^3$$

$$y = x^5 \qquad \frac{dy}{dx} = 5x^4$$

$$y = x^{-1} \qquad \frac{dy}{dx} = -x^{-2}$$

$$y = \sqrt{x} \qquad \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$y = 11 \qquad \frac{dy}{dx} = 0$$

Answer 19

$$y = 4x^3 + 17x^2 + 2x \qquad \frac{dy}{dx} = 12x^2 + 34x + 2$$

$$y = x^5 + 4x^3 + 6x^2 \qquad \frac{dy}{dx} = 5x^4 + 12x^2 + 12x$$

$$y = -x^2 + 3x + 10 \qquad \frac{dy}{dx} = -2x + 3$$

$$y = 3x^2 - 8x - 1 \qquad \frac{dy}{dx} = 6x - 8$$

$$y = -5x^3 + 10x^2 - 6x \qquad \frac{dy}{dx} = -15x^2 + 20x - 6$$

Answer 20

Showing my work, step-by-step:

$$x = x_o + v_o t + \frac{1}{2}at^2$$

$$\frac{d}{dt}x = \frac{d}{dt}x_o + \frac{d}{dt}v_o t + \frac{d}{dt}\frac{1}{2}at^2$$

$$\frac{d}{dt}x = 0 + v_o + at$$

$$\frac{dx}{dt} = v_o + at$$

Where,
 $\frac{dx}{dt}$ represents velocity.

Answer 21

$$U = \frac{1}{2}CV^2$$

Answer 22

$$U = \frac{1}{2}LI^2$$

$$V_{out} = -\frac{1}{RC} \int_{t_0}^{t_f} V_{in} dt + V_0$$

$$V_{out} = -\left(\frac{1}{(270 \times 10^3 \Omega)(0.33 \times 10^{-6} \text{ F})}\right) \left(\int_0^{0.5} 1.5 dt\right) + 0 \text{ V}$$

$$V_{out} = -\left(\frac{1}{0.0891 \text{ s}}\right) (0.75 \text{ V} \cdot \text{s}) + 0 \text{ V}$$

$$V_{out} = -8.418 \text{ V}$$

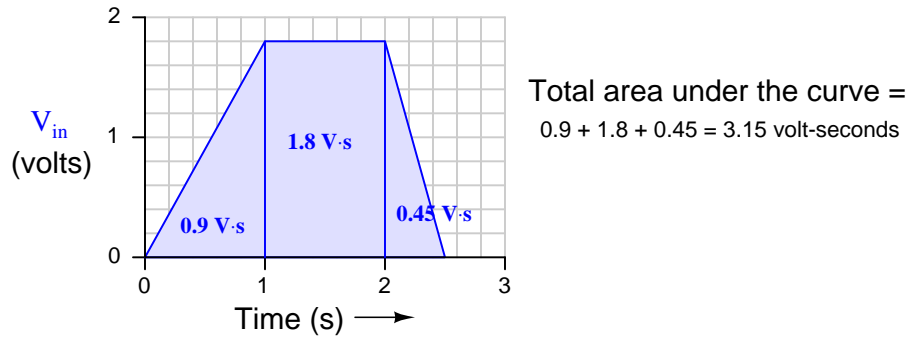
Calculating the time constant for this integrator circuit:

$$\tau_i = RC = (270 \times 10^3 \Omega)(0.33 \times 10^{-6} \text{ F}) = 0.0891 \text{ seconds}$$

$$V_{out} = -\frac{1}{RC} \int_{t_0}^{t_f} V_{in} dt + V_0$$

$$V_{out} = -\left(\frac{1}{(100 \times 10^3 \Omega)(2.2 \times 10^{-6} \text{ F})}\right) \left(\int_0^{2.5} V_{in} dt\right) + 8 \text{ V}$$

At this point we need to evaluate the integral in order to proceed much further. Since V_{in} is not a constant, and we have no means to symbolically integrate the input voltage function, we must find the integral value graphically. Recalling that the graphical meaning of integration is the geometric *area* encompassed by the function, all we need to do is calculate the area of the trapezoid:



Note the units of measurement used to express the integral: *volt-seconds*, because the vertical dimension is expressed in units of *volts* and the horizontal dimension is expressed in units of *seconds* and integration involves *multiplication* of units.

$$V_{out} = -\left(\frac{1}{0.22 \text{ s}}\right) (3.15 \text{ V} \cdot \text{s}) + 8 \text{ V}$$

$$V_{out} = -14.318 \text{ V} + 8 \text{ V}$$

$$V_{out} = -6.318 \text{ V}$$

Calculating the time constant for this integrator circuit:

$$\tau_i = RC = (100 \times 10^3 \Omega)(2.2 \times 10^{-6} \text{ F}) = 0.22 \text{ seconds}$$

At 1:30 PM, the level is increasing at a rate ($\frac{dV}{dt}$) of 5 gallons per hour, which is equivalent to 0.083 GPM. At that time the outgoing flow rate (FT-240) registers 35 GPM. Therefore, the incoming flow rate must be 0.083 GPM greater than FT-240, which is 35.083 GPM.

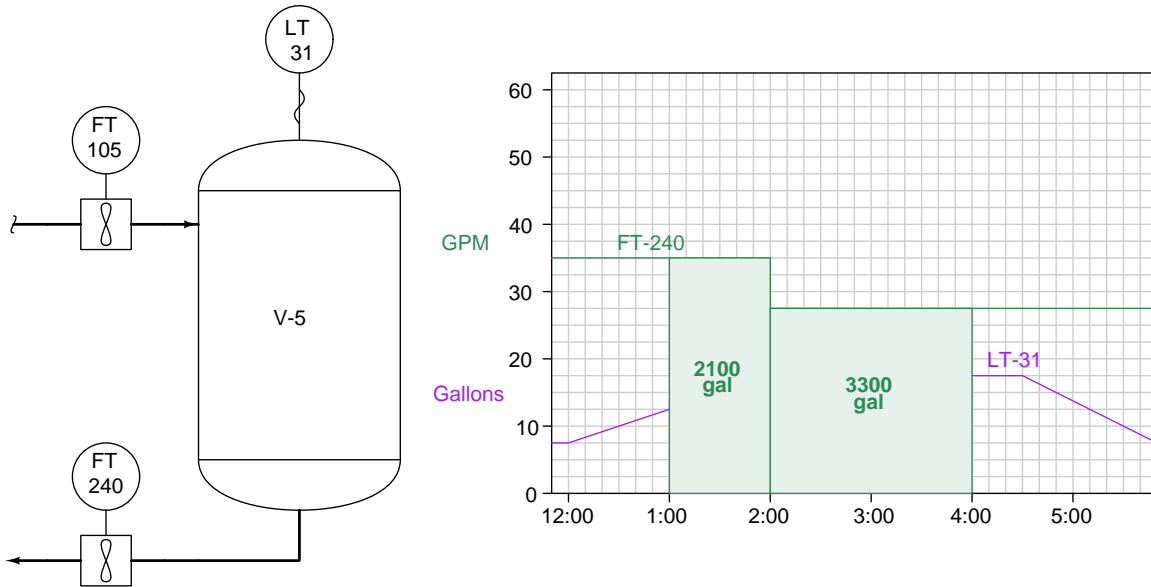
Q_{in} @ 3:45 PM must be equal to Q_{out} because the level (LT-31) is holding steady. Therefore, $Q_{in} = 27.5$ GPM at 3:45 PM.

What this means is that the incoming flow decreased at the same time as the outgoing flow decreased (both at 2:00 PM). Between Noon and 2:00 PM Q_{in} was 35.083 GPM and Q_{out} was 35 GPM, but then both flow rates stepped down to 27.5 GPM at 2:00 PM.

Here we are asked to calculate a total volume given flow rate (in gallons per minute) and time. This involves multiplication (so that minutes of time will cancel out the "minutes" in GPM to yield an answer in gallons), which means the appropriate calculus function is *integration*. Specifically, we need to integrate the flow rate of FT-240 over the time interval of 1:00 PM to 4:00 PM:

$$V_{discharged} = \int_{1:00 \text{ PM}}^{4:00 \text{ PM}} Q_{FT-240} dt$$

This integral represents the area beneath the FT-240 flow function between 1:00 PM and 4:00 PM on the trend graph, represented by the two shaded rectangles below:



The first rectangle is 35 GPM high and 60 minutes wide, yielding an area of 2100 gallons:

$$\left(\frac{35 \text{ gal}}{\text{min}}\right) \left(\frac{60 \text{ min}}{1}\right) = 2100 \text{ gal}$$

The second rectangle is 27.5 GPM high and 120 minutes wide, yielding an area of 3300 gallons:

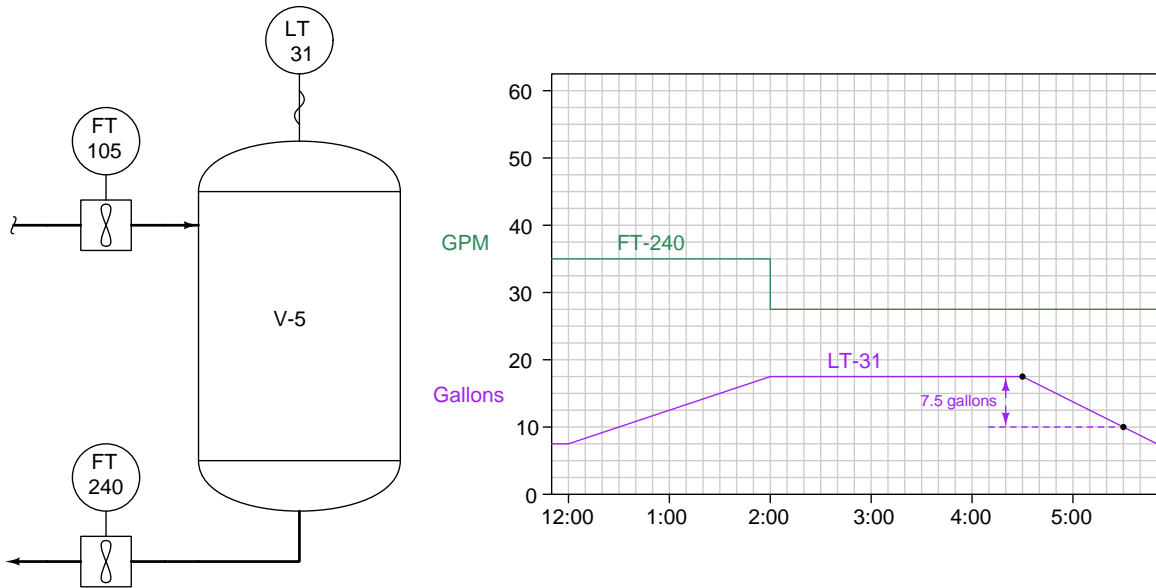
$$\left(\frac{27.5 \text{ gal}}{\text{min}}\right) \left(\frac{120 \text{ min}}{1}\right) = 3300 \text{ gal}$$

Together, the total area of these two rectangles is 5400 gallons, which is the value of our integral, and therefore the total quantity of liquid discharged from the vessel between 1:00 PM and 4:00 PM.

$V_{discharged}$ between 1:00 PM and 4:00 PM = **5400** gallons

Answer 27

This question is designed to probe your critical thinking, because there is absolutely no calculus involved in the answer! Level transmitter LT-31 already measures the amount of liquid stored in the vessel, so calculating volume lost between any two points in time is simply a matter of subtracting those LT-31 values at those times:



Since the vessel holds 17.5 gallons of liquid at 4:30 PM and holds 10 gallons of liquid at 5:30 PM, the amount of liquid lost from the vessel between those times is 7.5 gallons:

$$V_{lost} \text{ between 4:30 PM and 5:30 PM} = \underline{7.5} \text{ gallons}$$

We are asked to find the number of *gallons* inside the vessel given flow rates in *gallons per hour* and time in *hours*. Therefore, the mathematical operation we must employ is multiplication (so that “hours” cancels out to leave “gallons”) and that means *integration*.

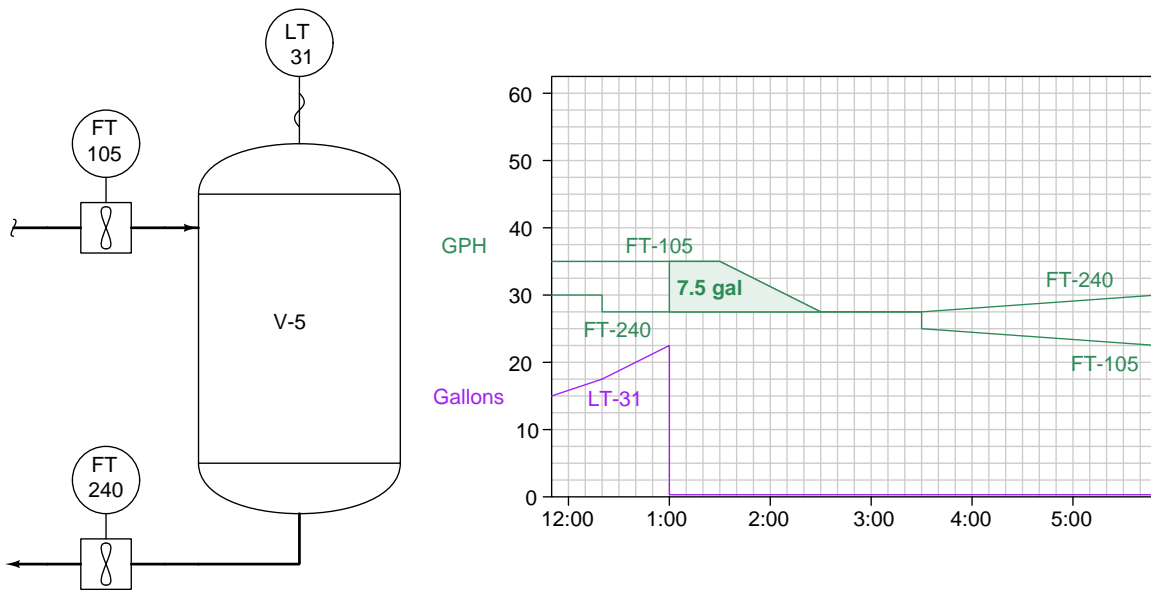
The vessel’s accumulated volume rises or falls according to the difference between the incoming and outgoing flow rates over time. Mathematically we may express this by the following integral:

$$V = \int_{t_0}^{t_f} (Q_{in} - Q_{out}) dt + V_0$$

The level transmitter stopped working at 1:00 PM, so the last known volume of liquid inside the tank was measured then: 22.5 gallons. This must then be our initial volume (V_0), with 1:00 PM being the lower limit of our integration interval. The upper limit of our integration interval must be 3:00 PM which is the time when we’re interested in the tank’s liquid volume. Therefore:

$$V_{3:00} = \int_{1:00}^{3:00} (Q_{in} - Q_{out}) dt + 22.5 \text{ gal}$$

Evaluating this integral graphically, we end up with the following shaded area between the two flowmeter trendlines:



The area of this trapezoid (calculated as the area of a 3x3 square and a 3x6 triangle) is $(7.5 \text{ gal/hr})(0.5 \text{ hr}) + (0.5)(7.5 \text{ gal/hr})(1 \text{ hr}) = 7.5 \text{ gallons}$. Thus:

$$V_{3:00} = \int_{1:00}^{3:00} (Q_{in} - Q_{out}) dt + 22.5 \text{ gal}$$

$$V_{3:00} = 7.5 \text{ gal} + 22.5 \text{ gal} = 30 \text{ gal}$$

Vessel V-5 therefore contains 30 gallons of liquid at 3:00 PM.